A computer-assisted upper bound for the de Bruijn–Newman constant

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Abstract. We construct an explicit piecewise-affine barrier for the squared imaginary parts of the zeros of the heat deformation of Riemann's xi function. The resulting upper bound for the de Bruijn–Newman constant is

$$B = \frac{3885632262767861213460393068710302759}{24646172707879668706230182733520000000} = 0.157656619095490606768\ldots < 0.158.$$

The argument combines classical zero dynamics with a three-probe source estimate, a density lower bound, and a bound for a logarithmic-derivative jet. Its finite barrier certificate contains 12,666 rational rows. Supporting computations use rigorous ball arithmetic; a second arithmetic library checks the elementary profile inequalities. A Lean 4 companion verifies the barrier comparison and zero dynamics, conditional on explicitly stated analytic inputs. The finite verification of RH used in the argument is the published theorem of Platt and Trudgian.

1. Introduction

For real t, let

$$H_t(z) = \int_0^\infty e^{tu^2}\Phi(u)\cos(zu)\,du,$$
$$\Phi(u) = \sum_{n\geq 1}(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u})e^{-\pi n^2 e^{4u}}.$$

This is the normalization of Polymath [Po19], in which

$$H_0(z) = \frac{1}{8}\,\xi\Big(\frac{1+iz}{2}\Big), \qquad \partial_t H_t = -\partial_z^2 H_t.$$

Here $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, continued to an entire function. The de Bruijn–Newman constant is

$$\Lambda = \inf\{t \in \mathbb{R} : H_t \text{ has only real zeros}\}.$$

De Bruijn proved $\Lambda \leq 1/2$ [dB50]; Newman proved finiteness from below and conjectured $\Lambda \geq 0$ [Ne76]. Rodgers and Tao proved this conjecture [RT20]. Thus RH is equivalent to $\Lambda = 0$.

Polymath obtained $\Lambda \leq 0.22$ [Po19]. Platt and Trudgian's rigorous verification of RH to height $3\cdot 10^{12}$ gives $\Lambda \leq 0.2$ [PT21, Corollary 2]. Gomila has made public a computer-assisted argument for $\Lambda \leq 0.1787854$, described by its author as not yet peer reviewed [Go26]. That argument instantiates Polymath's upper-bound criterion. The present argument uses the same classical heat-flow framework and finite-RH input, with density and jet estimates providing additional contraction of a moving barrier. No claim of priority over concurrent work is needed for the proof.

Theorem 1.1. In this normalization,

$$\Lambda \le B = \frac{388563226276786121346039306871030 2759}{246461727078796687062301827335200000 00}.$$

The mathematical argument is given below and in the two analytic supplements. The distinction between its conventional proof and the scope of its Lean formalization is recorded in §8. The theorem is a positive upper bound; it does not prove RH.

2. Notation and classical inputs

Write $z = x + iy$. The corresponding zeta coordinate is $s = (1 - y)/2 + ix/2$. Set

$$u(z,t) = \frac{H_t'(z)}{H_t(z)}, \qquad S(x,Y,t) = -\operatorname{Im} u(x+iY,t),$$
$$\Omega(x) = \frac{1}{4}\log\frac{x}{4\pi}, \quad J(x,Y,t) = |u_z| + (\operatorname{Re} u + \pi/8)^2 + (S-\Omega)^2,$$

where the quantities in J are evaluated at $x + iY$. Also $S_Y = -\operatorname{Re} u_z$. The fixed spatial cutoffs are

$$X = 5999347341500, \quad X_L = X - 10^6, \quad X_e = 5900000000000, \qquad \Omega_L = \frac{6722911}{10^6}.$$

In particular, $\Omega(x) > \Omega_L$ for $x \ge X$. This elementary inequality is proved in Lean and checked independently by interval arithmetic.

For $Y > b \ge 0$, define the conjugate-pair kernel

$$K_Y(d,b) = \frac{Y-b}{d^2 + (Y-b)^2} + \frac{Y+b}{d^2 + (Y+b)^2}.$$

When Y lies above all zeros, pairing conjugate zeros gives $S(x,Y,t) = \sum_{\text{pairs}} K_Y(x - \operatorname{Re}\rho, |\operatorname{Im}\rho|)$. Multiplicities are retained; real zeros have half-pair weight.

We use the following classical facts [Po19, §§1, 3]. Each H_t is real, even, entire, and not identically zero. For $t \ge 0$, its zeros lie in $|\operatorname{Im} z| \le 1$. Zeros persist locally under small changes of time, with multiplicity counted. A simple zero has a differentiable branch with velocity

$$\rho'(t) = \frac{H_t''(\rho(t))}{H_t'(\rho(t))}.$$

The regularized Hadamard identity at a simple zero is

$$\frac{H_t''(\rho)}{H_t'(\rho)} = 2\left\{\frac{1}{\rho} + \sum_{\zeta\ne\rho}\left(\frac{1}{\rho-\zeta} + \frac{1}{\zeta}\right)\right\}. \tag{2.1}$$

The regularized sum is absolutely convergent. Equivalently, one may use a symmetrically paired reciprocal-zero sum. Absolute convergence of the unpaired complex series is not asserted. The imaginary parts and pair-kernel sums used below are absolutely summable.

At a zero z_0 of multiplicity $m \ge 2$ at time t_0, backward evolution gives zeros

$$z_0 + i\sqrt{2\epsilon}\,\lambda_j + O(\epsilon) \quad \text{of } H_{t_0-\epsilon}, \tag{2.2}$$

where the λ_j are the real roots of the probabilists' Hermite polynomial He_m. One such zero has imaginary part $\operatorname{Im} z_0 + c\sqrt{\epsilon} + O(\epsilon)$ with $c > 0$. The formal proof establishes the needed height increase directly.

De Bruijn's strip theorem states that a zero strip of half-width h at time t becomes a real spectrum at time $t + h^2/2$ [dB50; Po19, Theorem 3.2].

3. The two barriers

An affine row is (t_L, t_R, q_L, q_R) with $t_L < t_R$ and $q_R < q_L$. Its value and positive speed are

$$q(t) = q_L + \frac{q_R - q_L}{t_R - t_L}(t - t_L), \qquad w = \frac{q_L - q_R}{t_R - t_L}.$$

Adjacent rows join exactly in time and squared height. We use a reference barrier q_{ref} on $[0, T_*]$ and a main barrier Q on $[3/50, T]$. Their exact rational data are in the compact barrier certificate. A third trace q_M specifies the domain of the density and jet estimates.

Trace	Interval	Rows	Purpose
q_M	$[0, T_M]$	1,620	Density and jet domain
q_{ref}	$[0, T_*]$	7,849	Strict entry for Q
Q	$[3/50, T]$	4,817	Final bound

The horizons are

$$\begin{aligned} T_M &= \frac{10724023263453313712965415492196719802951}{66207211195936838001560220771250000000000}, \\ T_* &= \frac{3727212594484883717859635359632643731427893093}{23745856366798857962956857522257275000000000000}, \\ T &= \frac{3854824546883011627577605340293402759}{24646172707879668706230182733520000000}. \end{aligned}$$

They satisfy $3/50 < T < T_* < T_M < 1/5$. The endpoint and entry identities are

$$\begin{aligned} q_{\mathrm{ref}}(0) &= (1000001/1000000)^2 > 1, & q_{\mathrm{ref}}(T_*) &= 1/400, \\ Q(3/50) &= \frac{84648870770133}{200000000000000} = q_{\mathrm{ref}}(3/50) + 10^{-9}, & Q(T) &= 1/400. \end{aligned} \tag{3.1}$$

Both barriers are continuous, piecewise affine, strictly decreasing, and at least $1/400$.

Theorem 3.1 (reference comparison). For $0 \le t \le T_*$, every zero ρ of H_t satisfies $(\operatorname{Im} \rho)^2 < q_{\mathrm{ref}}(t)$.

Theorem 3.2 (main comparison). For $3/50 \le t \le T$, every zero ρ of H_t satisfies $(\operatorname{Im} \rho)^2 < Q(t)$.

Deduction of Theorem 1.1. At time T, all zeros lie in the strip of half-width $1/20$. De Bruijn's theorem gives a real spectrum at

$$T + \frac{(1/20)^2}{2} = T + \frac{1}{800} = B. \tag{3.2}$$

These are exact rational identities. The comparisons are proved in §7.

4. Supporting estimates

The following statements specify every analytic input beyond §2. The accompanying analytic and computational supplements provide their proofs and executable enclosures. Labels P2–P8 are retained to connect the paper to the formal interfaces.

P2 (finite head). For $t \in [0, 1/5]$, every zero of H_t with $|\operatorname{Re} z| \leq X$ is real. The finite-RH input is [PT21, Theorem 1], since $X/2 < 3 \cdot 10^{12}$. Boundary non-vanishing is established by a direct 800,000-cell interval computation on $(t, y) \in [0, 1/5] \times [0, 1]$ at $x = X$. A zero-counting homotopy and forward Hermite splitting give the finite-rectangle conclusion. This computation does not reverify the published finite-RH theorem.

P3a (confinement). For every $0 < \tau \leq T' \leq 1/5$, nonreal zeros of H_t have uniformly bounded real part for $t \in [\tau, T']$. Indeed, [Po19, Theorem 1.5(i)] gives an absolute $C > 0$ such that zeros with $x \geq \exp(C/t)$ are real. Taking $R = \exp(C/\tau)$ and using evenness proves the assertion. No numerical value of R is required.

P3b (effective approximation). The normalized approximation $H_t/B_t = f_N + E$ is used on $x \geq X_e$, $t \in [0, 1/5]$, $0 \leq y \leq 7$. The analytic supplement defines every term and proves the extended domain and errors, including the vertical Stirling remainder, Gaussian tails, and cutoff changes. This domain is not an unqualified application of the narrower numerical domain in [Po19, Theorem 1.3]. Its consequences include

$$S(x, 5, t) > 6.59 \quad (x \geq X_L), \qquad |u(x + 5i, t)| < 10 \quad (X_e + 1 \leq x \leq X).$$

P4 (density). Define

$$\begin{aligned} U(Y, t) &= 10^9 + \sum_{j=1}^{3} a_j e^{\lambda_j^2 t} \cosh(\lambda_j Y), \\ (a_1, a_2, a_3) &= (1758974, 2464729, 302096), \\ (\lambda_1, \lambda_2, \lambda_3) &= (19/4, 97/20, 131/20), \\ f_0(Y, t) &= U_Y/U, \qquad \delta(t) = e^{90t}/10^{14}, \qquad f_M = f_0 - \delta. \end{aligned}$$

For $x \geq X$, $0 \leq t \leq T_M$, and $\sqrt{q_M(t)} < Y \leq 5$, $H_t(x + iY) \neq 0$ and $S(x, Y, t) \geq f_M(Y, t)$. The proof couples the vertical Burgers comparison with the wall q_M and checks initial, lower, side, top, and infinity boundaries. At fixed $Y > 0$, f_0 is nondecreasing in t by a covariance identity; also $\delta(T_M) < 10^{-5}$. Monotonicity of f_M is not needed.

P5 (jet). Set

$$C(Y, t) = e^{t/10^5} \left(\frac{8}{25} + \frac{1130000000000}{U(Y, t)} \right), \qquad b(t) = \sqrt{q_M(t)} + \frac{3}{5}.$$

For $x \geq X$, $0 \leq t \leq T_M$, and $b(t) \leq Y \leq 5$, one has $J(x, Y, t) \leq C(Y, t)$. The proof supplies the differential inequality for J, the initial and exterior boundary estimates, and all 162 boxes of the lower moving boundary. The statement is confined to this finite height band.

P7 (three-probe source). Set

$$V(x, h, t) = \frac{15}{14} S(x, 3h, t) - \frac{16}{21} S(x, 4h, t) + \frac{1}{6} S(x, 5h, t).$$

For each of the 26 source boxes, $V(x, h, t) > L$ throughout its closed (t, h) rectangle and for every $x \geq X$. The exact boxes and floors are tabulated in the source-box supplement. The computation establishes all required non-vanishing directly and uses neither q_M nor the jet bound. The uniform reduction, infinite tails, and integer-cutoff changes are part of the proof.

P8 (profile values). Every reference row has a probe p and floor s with

$$s < f_0(p, t_L) - 10^{-5}. \tag{4.1}$$

Signed rows also carry c with

$$c \geq e^{t_R/10^5}\left(\frac{8}{25} + \frac{1130000000000}{U(p, t_L)}\right). \tag{4.2}$$

These inequalities are checked on all 7,849 reference rows with Arb at 512 bits and independently with mpmath intervals at 80 decimal digits. For $t \in [t_L, t_R]$, P4 gives

$$S(x, p, t) \geq f_0(p, t) - \delta(t) > f_0(p, t_L) - 10^{-5} > s. \tag{4.3}$$

Monotonicity of U and P5 similarly give $J(x, p, t) \leq c$ on a signed row. The main barrier uses exactly these field records on contained time cells.

5. Kernel inequalities and velocity bounds

Let $\rho = x + ih$, $h > 0$, be a simple zero of maximal imaginary part and write $q = h^2$. Separating its conjugate from (2.1) gives

$$h' = -\frac{1}{h} - 2\sum_{\text{other pairs}} K_h(d_j, b_j), \qquad q' = 2hh'. \tag{5.1}$$

All other pair heights satisfy $0 \leq b_j \leq h$. At the lower height h, the singular point $(d, b) = (0, h)$ is excluded; it would represent the distinguished zero itself.

Three rational-function inequalities hold on this domain:

1. If $p^2 \geq 5h^2$ and $p > 0$, then $K_h \geq (h/p)K_p$.
2. With $(c_1, c_2, c_3) = (15/14, -16/21, 1/6)$, $K_h \geq c_1K_{3h} + c_2K_{4h} + c_3K_{5h}$.
3. If $p^2 \geq 9h^2$ and $p > 0$, put

   $$\alpha = \frac{h(3p^2 - h^2)}{2p^3}, \qquad a = \frac{p(p^2 - h^2)}{3p^2 - h^2}.$$

   Then $K_h \geq \alpha(K_p - a\,\partial_p K_p)$.

These are proved by clearing positive denominators; the three-probe case also has an interpolation proof. Their exact polynomial proofs are formalized in the companion kernel module.

5.1. Source rows

The distinguished conjugate pair contributes $\sum_j c_j K_{(j+2)h}(0, h) = 7/(15h)$. The second kernel inequality and (5.1) therefore give

$$q' \leq -\frac{2}{15} - 4\sqrt{q}\,V. \tag{5.2}$$

Suppose a row lies in a source box:

$$t_a \leq t_L < t_R \leq t_b, \qquad h_{\mathrm{lo}}^2 \leq q_R < q_L \leq h_{\mathrm{hi}}^2, \qquad L > 0. \tag{5.3}$$

Its speed is strictly beaten whenever

$$w \le \frac{2}{15} \quad \text{or} \quad \left(w > \frac{2}{15} \text{ and } (w - \tfrac{2}{15})^2 < 16L^2 q_R \right). \tag{5.4}$$

Indeed, at contact $q \ge q_R$, so $q' \le -2/15 - 4L\sqrt{q_R} < -w$. The sign conditions in (5.4) are necessary when squaring.

5.2. Unsigned field rows

The first kernel inequality, with the distinguished pair separated, yields

$$q' \le -2 - \frac{4qS}{p} + \frac{8q}{p^2 - q} < -2 - qK_u(q), \qquad K_u(q) = \frac{4s}{p} - \frac{8}{p^2 - q}. \tag{5.5}$$

Here $S > s$ is supplied by (4.3). The function K_u is decreasing for $q < p^2$.

5.3. Signed field rows

Set $D = S - aS_Y$, where $a = p(p^2 - q)/(3p^2 - q)$. The third kernel inequality gives

$$q' \le -2 - 4h\alpha D + \frac{16q}{p^2 - q}. \tag{5.6}$$

From $J \le c$ and $S_Y = -\operatorname{Re} u_z$, $D \ge S - ac + a(S - \Omega)^2$. If $s < \Omega_L < \Omega$ and $2a(\Omega_L - s) \le 1$, then

$$S - ac + a(S - \Omega)^2 > s - aL_0, \qquad L_0 = c - (\Omega_L - s)^2. \tag{5.7}$$

To see this, write $u_0 = S - s \ge 0$, $v = \Omega - \Omega_L > 0$, and $d = \Omega_L - s > 0$. The difference is $u_0(1 - 2ad) + a(u_0 - v)^2 + 2adv > 0$.

Substitution in (5.6) gives

$$\begin{aligned} q' &< -2 - qK_s(q), \\ K_s(q) &= \frac{6s}{p} - 2L_0 + q\left(\frac{2L_0}{p^2} - \frac{2s}{p^3}\right) - \frac{16}{p^2 - q}. \end{aligned} \tag{5.8}$$

If $pL_0 \le s$, then $K_s'(q) < 0$. The coefficient $a(q)$ is decreasing, so its branch condition need only be checked at q_R.

For either field mode let $k = K(q_L) > 0$. If

$$w \le 2 + kq_R, \tag{5.9}$$

then at contact $q \in [q_R, q_L]$, $q' < -2 - qK(q) \le -2 - kq_R \le -w$. Equality in the rational speed gate is harmless because the analytic inequality is strict.

6. Exact certificate conditions

The compact certificate stores rational values, source-box indices, and references to field records. The exact checker verifies:

- proper rows, exact joins, prescribed endpoints, and $q_R \ge 1/400$;
- closed source-box containment and the strict speed condition (5.4);
- containment in each field record's time cell, of length at most $1/50000$;

- exact interpolation of the density wall at the row's left endpoint;
- $0 < p \le 5$, $p^2 > q_M(t_L)$, and $p^2 > q_L$;
- in unsigned mode, $5q_L \le p^2$;
- in signed mode, $p \ge 3/5$, $(p-3/5)^2 \ge q_M(t_L)$, $9q_L \le p^2$, $c > 0$, $\Omega_L > s$, $pL_0 \le s$, and $2a(q_R)(\Omega_L - s) \le 1$;
- positivity of the recomputed k and the speed condition (5.9).

The decreasing wall makes the left-endpoint domain checks valid throughout the row. The main barrier has 541 source, 2,772 signed field, and 1,504 unsigned field rows. The reference barrier has 6,317 signed and 1,532 unsigned field rows. Every source box is used.

The Python checker uses exact rational arithmetic and explicit failure checks. In Lean, equivalence between the executable row check and its propositional specification is proved. Kernel evaluation verifies every row, the chains, source catalog, and wall. A deterministic generator checks the correspondence between the compact JSON and generated Lean declarations.

The certificate is a witness. Its validity does not depend on how it was optimized, unused candidate rows, rejected numerical targets, or producer acceptance flags.

7. First-contact proof

We give a common argument for both barriers. For Q, the starting time is $t_0 = 3/50$. Theorem 3.1 and (3.1) give strict entry. For the reference barrier, the global strip gives strict entry at zero. Exact arithmetic shows that

$$\tau_0 = \frac{20000010}{238529173356019} > 0, \qquad q_{\mathrm{ref}}(t) > 1 \quad (0 \le t \le \tau_0).$$

Its first possible contact is therefore after τ_0, where confinement is uniform.

Let q_b be the barrier under consideration. Suppose a zero reaches or exceeds it, and define

$$\mathcal{F} = \{t : \exists z,\ H_t(z) = 0,\ (\operatorname{Im} z)^2 \ge q_b(t)\}.$$

Every such zero has $|\operatorname{Im} z| \ge 1/20 > 1/40$. P3a bounds its real part uniformly on the positive-time interval. Joint continuity of H and continuity of the barrier make the relevant zero set closed in a compact box. Thus $\mathcal{F}$ has a first time $t_1 > t_0$.

Local persistence excludes a strict overshoot: a zero strictly above the barrier would persist above it at earlier nearby times. Hence a highest zero $\rho_1 = x_1 + ih_1$ satisfies $h_1^2 = q_b(t_1)$, and all zeros lie at or below this height. P2 gives $|x_1| > X$. Evenness and conjugation allow $x_1 > X$ and $h_1 > 0$.

A multiple contact is impossible. By (2.2), backward evolution produces a zero at height $h_1 + c\sqrt{\epsilon} + O(\epsilon)$. The function $\sqrt{q_b}$ is Lipschitz because $q_b \ge 1/400$ and has finitely many affine pieces. Its earlier height is only $h_1 + O(\epsilon)$, contradicting first contact.

The contact zero is simple. Choose the row ending at t_1 if t_1 is a knot, so its slope is the left derivative. P7 and §5.1, or P4–P5–P8 and §§5.2–5.3, give

$$\left.\frac{d}{dt}(\operatorname{Im} \rho(t))^2\right|_{t=t_1} < q_b'(t_1^-).$$

But $(\operatorname{Im} \rho(t))^2 - q_b(t)$ is negative just before t_1 and zero at t_1, so its left derivative is nonnegative. This contradiction proves both comparisons.

The argument includes terminal times, changes of row type, and multiple zeros. It never differentiates an unattained supremum or evaluates a logarithmic derivative at a zero. Equation (3.2) completes Theorem 1.1.

8. Verification scope and reproducibility

The manuscript, source code, exact certificates, and reproduction instructions are maintained at github.com/stefangordon/dbn-upper-bound.

The classical inputs are cited published theorems. The new analytic estimates are conventional proofs in the supplements, with rigorous finite computations furnishing their explicit inequalities. The Lean companion verifies the deduction from stated analytic inputs, including the concrete heat integral, zero dynamics, kernel inequalities, barrier calculus, and first-contact argument. Its main theorem remains conditional on those inputs.

Every P8 endpoint gate is kernel-checked using a proved rational evaluator. Finite-head propagation is proved from the time-zero finite-RH input and vertical boundary non-vanishing; neither input is thereby proved.

An alternative interface assumes the residual estimate and nonzero normalizer of Lemma A.1, finite RH at time zero, the continuous lower bound $|f^{[N]}(X + iy)| > 1/500$, source floors, and density/jet comparisons. Lean derives the required confinement and finite head from these inputs. The continuous bound is not a Lean-checked grid: the polynomial computation and its Taylor-error bridge remain unformalized. Auxiliary zeta and finite heat-sum identities are proved, but the remaining analytic inputs are not. The published positive-time tail statement provides an alternative confinement input.

For the main theorem, Lean reports only its standard foundational axioms: propositional extensionality, classical choice, and quotient soundness. This report does not prove the theorem's explicit hypotheses.

The standalone numerical entry point is:

```
python3.12 -m venv .venv
.venv/bin/pip install -r requirements.txt
.venv/bin/python verify.py --regenerate
```

The supplied Dockerfile provides CPython 3.12, a C compiler, and FLINT headers. This command regenerates the boundary matrix and runs all numerical checks, including every boundary cell, source box, field estimate, and both P8 arithmetic backends. Late boundary boxes run in separate processes. The explicit quick mode omits the supporting computations.

To check the formal companion:

```
cd lean
lake exe cache get
lake build
lake env lean Audit.lean
```

Python packages and Lean dependencies are version-pinned; native tool versions are recorded by the runs. The release manifest identifies the published bytes; it is an integrity mechanism, not a proof of the mathematics.

The research and initial drafts were produced with AI assistance. Subsequent automated reviews independently reconstructed mathematical arguments and numerical evaluations. Such reviews are not external peer review. Responsibility for the submitted statements and their verification remains with the human authors.

Appendix A. Analytic estimates

This appendix proves the analytic estimates used as P3–P5 and supplies the whole-row analytic interpretation of P8. The finite-head theorem P2 and the three-probe source floors P7 are proved in Appendix C. The classical zero identities and first-contact argument are stated in the main text. The only non-elementary approximation input used here is the generic-real-part Riemann–Siegel expansion of Arias de Reyna, in the explicit form recorded in [Po19, Proposition 6.2 and equation (58)]. In particular, the restriction $0 \le y \le 1$ in [Po19, Theorem 1.3] is respected: the larger range below is derived from the generic-real-part expansion, with new error estimates.

Write $X_e = 5900000000000$, $X_L = 5999346341500$, $X = 5999347341500$, $t_m = 1/5$, $N_e = 685205$, and $N_0 = 690950$. All decimal constants in this appendix denote exact rationals. The accompanying programs use exact rational geometry and Arb ball arithmetic at 512 bits. Statements such as $a < b$ in a program mean that the entire computed ball for a lies strictly below the entire ball for b.

A.1. Normalization and the enlarged approximation

For $z = x + iy$, put $w = (1 - iz)/2$, $v = 1 - w$, and define

$$M_0(s) = \frac{s(s-1)}{16}\pi^{-s/2}\sqrt{2\pi}\exp\left[\left(\frac{s}{2}-\frac{1}{2}\right)\operatorname{Log}\frac{s}{2}-\frac{s}{2}\right],$$

$$\alpha(s) = \frac{1}{2s}+\frac{1}{s-1}+\frac{1}{2}\operatorname{Log}\frac{s}{2\pi}, \qquad M_t(s) = M_0(s)e^{t\alpha(s)^2/4},$$

$$B_t(z) = M_t(w), \qquad s_B = w + \frac{t}{2}\alpha(w),$$

$$s_A = v + \frac{t}{2}\alpha(v), \qquad \gamma = M_t(v)/M_t(w),$$

$$b_t(n) = e^{t\log^2 n/4}, \qquad P_N(s,t) = \sum_{n=1}^{N} b_t(n)n^{-s}.$$

The principal logarithms are holomorphic throughout the large-x regions used below. In particular B_t is nonzero there. Put $\mathcal{Q} = x/(4\pi)$, $L = \log\mathcal{Q}$, $Z = (L^2 + \pi^2/4)^{1/2}$, and $N = \lfloor\sqrt{\mathcal{Q} + t/16}\rfloor$. The holomorphic expression with a *fixed* integer cutoff is $f^{[N]} = P_N(s_B, t) + \gamma P_N(s_A, t)$.

Lemma A.1 (enlarged approximation). For every $x \ge X_e$, $0 \le t \le t_m$, and $0 \le y \le 7$,

$$\left|\frac{H_t(z)}{B_t(z)} - f^{[N]}(z)\right| \le E_{ab}(x) + E_c(x,t,y), \tag{A.1}$$

where

$$d(x) = \frac{t_m^2L^2/16 + 0.626}{x - 6.66},$$

$$E_{ab}(x) = 2.01\frac{1000}{999}\frac{1.001^{0.501}}{0.501}\mathcal{Q}^{0.2505}d(x),$$

$$V_7(x) = \frac{20Z + 200}{x - 40},$$

$$E_c(x,t,y) = \mathcal{Q}^{-(1+y)/4}\exp\left[-\frac{tL^2}{16} + \frac{2(3^y + 3^{-y})}{N-1} + 10^{-10} + V_7(x)\right].$$

Here $E_{ab}(x) < 3\cdot 10^{-9}$ and the total error in (A.1) is less than $1/500$. Both $E_{ab}(x)$ and $E_c(x,t,y)$ are nonincreasing in x, with the latter interpreted across the upward jumps of N.

Proof. We give the details needed to enlarge the original parameter range. Direct differentiation gives

$$\alpha' = -\frac{1}{2s^2} - \frac{1}{(s-1)^2} + \frac{1}{2s}.$$

For $s = \sigma + ix/2$, $|\sigma| \le 4$, elementary bounds for the logarithm and the two rational terms give

$$\begin{aligned} |\alpha(s) - L/2 - i\pi/4| &\le 7/x + 16/x^2 \le 8/(x-20), \\ |\alpha'(s)| &\le 1/x + 6/x^2 \le 1/(x-6), \\ |\alpha''(s)| &\le 2/x^2 + 24/x^3. \end{aligned} \tag{A.2}$$

The last two inequalities hold for every real σ. Conjugation gives the signs appropriate to negative imaginary parts. Taking real parts of the explicit formula also gives $\Re\alpha((1+y-ix)/2) \ge L/2 - 2/x^2$ for $y \ge 0$. The possibly negative rational term occurs only when $y < 1$ and has magnitude at most $2/x^2$; when $y \ge 1$ both rational real parts are nonnegative.

Let $\beta = \alpha + (t/2)\alpha\alpha'$. Since $Z \le L + \pi/2$, (A.2) gives

$$|\Re\beta - L/2| \le \frac{tL/4 + 9}{x - 20} < 10^{-11}.$$

Integrating horizontally between real parts $(1-y)/2$ and $(1+y)/2$, both in $[-3, 4]$, proves

$$e^{-10^{-11}y} Q^{-y/2} \le |\gamma| \le e^{10^{-11}y} Q^{-y/2}, \qquad s_A = \overline{s_B} + \kappa - y, \quad |\kappa| \le \frac{ty}{2(x-6)}. \tag{A.3}$$

The heat-convolution identities [Po19, equations (39)–(41)] express H_t as two sums of $r_{t,n}$ and two Riemann–Siegel remainders $R_{t,N}$. The following estimate for $r_{t,n}(\sigma + iT)$ is valid for every real σ, $T > 10$, and $0 < t \le 1/2$:

$$\left| \frac{r_{t,n}(\sigma + iT)}{M_t(\sigma + iT) b_t(n) n^{-\sigma - iT - t\alpha(\sigma + iT)/2}} - 1 \right| \le \exp\left[\frac{t^2 |\alpha(\sigma + iT) - \log n|^2/8 + t/4 + 1/6}{T - 3.33} \right] - 1. \tag{A.4}$$

For completeness, the uniform Stirling estimate that justifies the Gaussian contour in (A.4) is proved in A.2. For every real σ, $\Im\alpha(\sigma + iT) \ge -1/(2T) - 1/T$, since the logarithm has positive argument. The shift $+(t/2)(\alpha - \log n)$ therefore leaves imaginary part at least $T - 0.0375 > T - 0.08$. On that contour the gamma argument has imaginary part at least $(T - 0.08)/2$, giving the error $1/[6(T-2.08)] \le 1/[6(T-3.08)]$. Taylor's formula for $\log M_0$ and $|\alpha'| \le 1/[2(T-3.08)]$ then bound the remaining exponent by

$$\frac{tv_0^2/2 + t^2|\alpha - \log n|^2/8 + 1/6}{T - 3.08}.$$

Integration against $e^{-v_0^2}/\sqrt{\pi}$ yields (A.4): the Gaussian factor $(1 - t/[2(T-3.08)])^{-1/2}$ is at most $e^{t/[4(T-3.33)]}$. Thus the proof of [Po19, Proposition 6.1] applies with this stated repair, for arbitrary real σ.

For $n \le N$, (A.2) gives $|\alpha - \log n|^2 \le \frac{1}{4}\log^2(\mathcal{Q}/n^2) + 0.667$. The remaining numerator is less than 0.313 because $t_m^2(0.667)/8 + t_m/4 + 1/6 < 0.313$. Also $|\log(\mathcal{Q}/n^2)| \le L$ on this domain. Hence the multiplier in (A.4) is at most $e^{d(x)} - 1 \le 1000d(x)/999$. Combining the heat factors *before* bounding them gives $|b_t(n)n^{-s_B}| \le n^{-0.499}$. By (A.3) and the cutoff relation the reflected-to-first magnitude ratio at $n \le N$ is less than 1.01. Finally $\sum_{n\le N} n^{-0.499} \le N^{0.501}/0.501$ and $N \le 1.001\sqrt{\mathcal{Q}}$. These inequalities give E_{ab}.

For the remainders, put $T = x/2$, $T' = T + \pi t/8$ and $a = \sqrt{T'/(2\pi)}$. Section A.2 proves, uniformly for $-3 \le \sigma \le 4$,

$$R_{t,N}(\sigma + iT) = \omega_N e^{i\pi\sigma/4} e^{t\pi^2/64} M_0(iT')\{C_0(p) + \varepsilon_\sigma\}, \quad |\omega_N| = 1, \quad |C_0(p)| \le 1/2,$$

with

$$|\varepsilon_{(1+y)/2}| + |\varepsilon_{(1-y)/2}| \le \frac{2(3^y + 3^{-y})}{N-1} + 10^{-10}. \tag{A.5}$$

In normalizing by B_t, the displacement from iT' to $(1+y)/2 + iT$ has squared modulus less than 17. The Taylor and linear errors are bounded by $3\sqrt{17}/x + 17/[2(x-6)] < 22/(x-20)$. In $\log|M_0(iT')/M_0((1+y)/2 + iT)|$ the displacement contributes $-t\pi^2/32$. The other real leading terms are $t\pi^2/64$ from the remainder contour, and $-t\Re\alpha^2/4 = -tL^2/16 + t\pi^2/64+$ error from M_t. The three π^2 terms cancel. The remaining positive error is at most

$$\frac{22 + 2tZ}{x - 20} + \frac{16t}{(x-20)^2} \le V_7(x).$$

The two C_0 contributions have total modulus at most one; use $1 + r \le e^r$ on their combined error (A.5). This gives E_c.

All endpoint reductions are monotonic reductions. For E_{ab} its logarithmic derivative multiplied by x is less than $0.2505 + 2/L - 1 < 0$. In E_c, the power and heat factors decrease, N is nondecreasing, and V_7 decreases since $xZ'(x) \le 1$ and its numerator is greater than 200. The scalar program `check_h7.py` proves $E_{ab}(X_e) < 1.389336 \cdot 10^{-9}$ and the whole-domain envelope $3\cdot 10^{-9} + \mathcal{Q}_e^{-1/4}\exp(2(3^7+3^{-7})/(N_e - 1) + 10^{-10} + V_7(X_e)) < 0.001215803$. The argument so far is for $t > 0$. For each fixed x, y, the actual cutoff is right-constant at $t = 0$, including an integral square-root value. Dominated convergence in the defining heat integral, using $e^{t_m u^2 + 7u}|\Phi(u)|$, passes (A.1) to $t = 0$. □

A.2. Uniform Stirling and the Gaussian Riemann–Siegel remainder

Lemma A.2 (vertical Stirling bound). On $\mathbb{C}\setminus(-\infty, 0]$, let $\theta(z) = \log\Gamma(z) - [(z - 1/2)\operatorname{Log} z - z + \frac{1}{2}\log(2\pi)]$, using the branch real on the positive axis. If $v_0 = |\Im z| > 1$, then

$$|\theta(z)| < \frac{1}{12(v_0 - 1)}. \tag{A.6}$$

Proof. Two-term Euler–Maclaurin, first for $\Re z > 0$, gives

$$\theta(z) = \frac{1}{12z} - \frac{1}{360z^3} - \frac{1}{4}\int_0^\infty \frac{B_4(\{u\})}{(z+u)^4}\,du, \quad B_4(r) = r^4 - 2r^3 + r^2 - \frac{1}{30}.$$

The integral is locally uniformly convergent on the slit plane, so the identity theorem extends the formula to that whole plane. This continuation retains the contribution present near the negative real axis. Since $|B_4(r)| \le 1/30$ on $[0,1]$ and $\int_{\mathbb{R}}(u^2+v_0^2)^{-2}\,du = \pi/(2v_0^3)$,

$$|\theta(z)| \le \frac{1}{12v_0} + \frac{1/360+\pi/240}{v_0^3} < \frac{1}{12(v_0-1)}.$$

The last inequality follows from $12(1/360+\pi/240)(v_0-1) < v_0^2$. $\square$

This bound is used in place of the unrestricted $|z|$-dependent estimate in [Po19, Lemma 5.1(v)]. A bound that decreases with $|\Re z|$ at fixed nonzero $\Im z$ cannot discard the reflection contribution near the negative real axis.

We now prove (A.5). In the contour for $R_{t,N}$, write $U = \sigma + \sqrt{t}V$ with density $e^{-V^2}/\sqrt{\pi}$, and put $A_g = T-3$, $D = 1 - t/A_g$. The gamma error from (A.6), the Taylor error for $M_0(U+iT')$, and the linear error in $\alpha(iT')$ have combined modulus

$$\frac{U^2+6|U|+2/3}{4A_g} \le \frac{U^2+1}{A_g}. \tag{A.7}$$

The 2/3 in (A.7) is necessary: the gamma bound is $1/[6(T'-2)]$. For $|\sigma| \le 4$, Gaussian completion gives

$$\begin{aligned}
\mathbb{E}e^{(U^2+1)/A_g} - 1 &\le D^{-1/2}e^{17/A_g+16t/(A_g^2 D)} - 1 < 6\cdot 10^{-12},\\
0.2\,\mathbb{E}(9^U e^{(U^2+1)/A_g}) &\le 0.2\,9^\sigma D^{-1/2} e^{17/A_g + t(\log 3 + 4/A_g)^2/D} < 0.26\,9^\sigma,\\
0.029\,\mathbb{E}(2^{-U}e^{(U^2+1)/A_g}) &\le 0.029\,2^{-\sigma}D^{-1/2}e^{17/A_g+t(\log 2/2+4/A_g)^2/D} < 0.030\,2^{-\sigma}.
\end{aligned} \tag{A.8}$$

Every positive factor is largest at $x = X_e, t = t_m$; `check_h7.py` verifies these endpoint inequalities. The coefficient 17 replaces the smaller-domain value 5.

Here are the explicit coefficient and tail bounds used with (A.8). The expansion in [Po19, Proposition 6.2] has coefficients $C_k(p,U)$ and remainder RS_K satisfying, with $\kappa_0 = ((3-2\log 2)\pi)^{-1/2}$ and $P_0 = \sqrt{2}/(4\pi^2)$,

$$\begin{aligned}
U \ge 0: &\quad |C_1|/a + |RS_1| \le 0.2\,9^U/a + 0.173\,2^{3U/2}/a^2 \le 0.2\,9^U/(a-0.865),\\
U < 0: &\quad |C_k|/a^k \le P_0\,2^{-U}\Gamma(k/2)(\kappa_0/a)^k.
\end{aligned}$$

Choose the measurable truncation $K=1$ when $U\ge 0$, and $K = \max(100, \lfloor -U\rfloor + 3)$ otherwise. Thus $K+U\ge 2$ whenever the negative-real-part remainder requires it. Separating even and odd indices shows

$$a\sum_{k=1}^{100} P_0\Gamma(k/2)(\kappa_0/a)^k \le P_0\frac{\sqrt{\pi}\kappa_0 + \kappa_0^2/a}{1-50\kappa_0^2/a^2} < 0.029. \tag{A.9}$$

For $K = 100$, retain RS_{100} separately. The elementary bound $\Gamma(101/2) \le 101^{101}$ and $1.1(101)/N_e < 10^{-3}$, together with (A.8) and $2^{-\sigma} \le 8$, give a Gaussian expectation less than 10^{-301}. All other indices beyond 100, including RS_K at its distinct index $k = K+1$, are bounded by $\frac{1}{2}2^{-U}1.1^k\Gamma(k/2)/a^k$ and satisfy $k \le -U+4$. Tonelli's theorem applies to these positive majorants. With $r = U-\sigma \le 7-k$ and $U^2+1 \le 2r^2+33$, their total is at most

$$8\sqrt{t/\pi}\,e^{33/A_g}\sum_{k\ge 100}\frac{1.1^k\Gamma(k/2)}{(k-8)a^k}\exp\left[-\frac{(k-7)^2}{2t} + (k-7)\log 2\right]. \tag{A.10}$$

The prefactor is less than 4. Using $\Gamma(k/2) \le k^k$, $\log k \le k/4$, $\log 1.1 < 0.1$, and $\log 2 < 1$, the logarithm of each numerator in (A.10) is at most $-3k^2/4 + 15.1k - 49 \le -k^2/2$ for $k \ge 100$. Thus (A.10) is less than $4e^{-5000} < 10^{-100}$. This accounts for the entire negative Gaussian tail, rather than a bounded range of its real parts.

The two applied real parts are $(1+y)/2$ and $(1-y)/2$. Their $0.26\,9^{\sigma}$ terms sum to $0.78(3^y+3^{-y})$, and $2^{-1/2}(2^{y/2} + 2^{-y/2}) \le 3^y + 3^{-y}$. The denominators $a - 0.865$ and $a - 0.353$ exceed $N - 1$. The remaining total $6 \cdot 10^{-12} + 2 \cdot 10^{-100} + 2 \cdot 10^{-301}$ is less than 10^{-10}. Including the two factors $|C_0| \le 1/2$ therefore gives (A.5).

A.3. Full disks, the top boundary, and infinity

Lemma A.3 (cutoff correction on a disk). Fix a center $x_0 + iY$, a time $0 \le t \le t_m$, and a disk of radius $0 < r \le 1/4$ lying inside $x \ge X_e$, $0 \le y \le 7$. Suppose every actual cutoff in the disk is at least $n_* \ge N_e$. If $0 \le y_d \le y \le y_u \le 7$ on the disk, let

$$\ell_* = \log(1 - t_m/(16n_*^2)), \qquad c_* = -t_m\ell_*/4,$$
$$k_* = \frac{t_m}{2(4\pi n_*^2 - \pi t_m/4 - 6)},$$
$$C_1 = \exp\left[(-(1+y_d)/2 + t_m/X_e^2 + c_*)\log n_* - t\log^2 n_*/4 + \frac{t_m \log(n_*+1)}{2n_*}\right],$$
$$C_2 = e^{10^{-11}y_u - y_u\ell_*/2}(1+1/n_*)^{y_u} e^{k_* y_u \log(n_*+1)}.$$

If N_c is the center cutoff, the holomorphic remainder $H_t/B_t - f^{[N_c]}$ is bounded on the entire disk by $3 \cdot 10^{-9} + \sup E_c + C_1(1 + C_2)$. Its jth derivative at the center is bounded by $j!r^{-j}$ times this disk bound, for $j = 1, 2$.

Proof. Two points in the disk differ in x by at most $1/2$, so their cutoffs differ by at most one. At a point of actual cutoff m, the changed index is $n = m$ or $m + 1$. Then $\mathcal{Q} \ge m^2 e^{\ell_*}$, $0 \le \log n - \log m \le 1/m$, and

$$\log^2 n - 2\log m \log n \le -\log^2 m + 2\log(m+1)/m.$$

Combining the negative coefficient of $\log n$ with c_* before taking endpoints gives C_1. Equation (A.3), $n/m \le 1 + 1/m$, and the decreasing function $\log(m+1)/(4\pi m^2 - \pi t_m/4 - 6)$ give C_2 for the second branch. The other combined factors decrease in m. Both branches are included. Cauchy's theorem applies to the fixed-N_c holomorphic remainder, not to the discontinuous actual-cutoff formula. At $t = 0$ (A.1) already holds on the full disk, so the same Cauchy argument applies. □

Replacing 10^{-11} by $1/50$ in C_2 is a permissible larger bound used in the late-boundary evaluator. For time boxes, replace the negative t exponent by the left time endpoint and the other positive quantities by their upper endpoints. The source error itself uses the disk's smallest y in its negative power and its largest y in $3^y + 3^{-y}$.

Lemma A.4 (top and head-center bounds). For $x \ge X_L$ and $0 \le t \le 1/5$, $H_t(x + 5i) \ne 0$ and $S(x, 5, t) > 6.59$. For $X_e + 1 \le x \le X$, also $|H_t'/H_t(x + 5i)| < 10$. Finally, for $x \ge X_L$ and $0 \le t \le 1/5$,

$$J(x, 5, t) < 0.309518 < 8/25. \tag{A.11}$$

Proof. Lemma A.3 with $Y = 5, r = 1/4, n_* = N_e$ gives a full disk error less than 10^{-8} and a center derivative error less than 10^{-7}. At height 5, combine the heat exponents to obtain ordinary weights $n^{-\rho}$, $\rho = 3 - t_m/X_e^2 > 2$. The first sum differs from 1 by at most $\zeta(\rho) - 1$, and its derivative has modulus at most $r_1(-\zeta'(\rho))$, $r_1 = 500001/10^6$. Retaining $(n/N)^y$ in the reflected sum bounds its value and derivative by $1.01N_e^{1-\rho} < 10^{-11}$ and $1.01N_e^{1-\rho}(2\log(N_e+1)+1) < 10^{-9}$. The power-log expression decreases for $N \geq N_e$; no unbounded logarithm has been replaced by a finite endpoint without its decaying factor.

For $F = H_t/B_t$ this proves

$$|F| \geq m = 2 - \zeta(\rho) - 10^{-11} - 10^{-8} > 0.79,$$
$$|F'/F| \leq K_3 = \frac{r_1(-\zeta'(\rho)) + 10^{-9} + 10^{-7}}{m} < 0.124148480.$$

The archimedean density loss is less than 10^{-10}. Thus $S(x,5,t) \geq \frac{1}{4}\log(X_L/(4\pi)) - 10^{-10} - K_3 > 6.598763368 > 6.59$. Taking the modulus of B'/B on the finite slab gives $|u| < 6.858519782 < 10$ there.

For (A.11), use the outer radius-one disk about $x + 5i$, with heights in $[4,6]$. At each point use an inner source disk of radius $1/4$, lying in $[15/4, 25/4] \subset [0,7]$. The same calculation with $\rho = 5/2 - t_m/X_e^2$ bounds $|F'/F|$ on the outer disk by $K_2 < 0.294104432$. This is a holomorphic function there: H_t has no zeros at these heights, and B_t is nonzero. Cauchy gives $|(F'/F)'(x+5i)| \leq K_2$. Direct differentiation of β gives archimedean errors $e_1, e_2 < 10^{-10}$ for u and u_z, respectively. Consequently $J \leq K_2 + e_2 + (K_3 + e_1)^2 < 0.309518$. All the stated numerical inequalities are reproduced by `check_h7.py` and `check_fixed_c.py`. □

Two distinct uniformity assertions are needed in the comparison proofs. For a compact positive-time interval and a compact y-band in $(0,7)$, combine the heat factors using $\log N \leq L/2 + O(1/\mathcal{Q})$. Each finite sum is bounded by an ordinary power majorant whose exponent tends to infinity with $\log x$. The first sum tends to 1, the reflected contribution tends to 0, and the actual x-dependent error in (A.1) tends to 0. On slightly larger disks this proves $H_t/B_t \to 1$, with two Cauchy derivatives, uniformly. Thus

$$S(x,y,t) \to \infty, \qquad J(x,y,t) \to 0 \tag{A.12}$$

uniformly on these positive-time bands.

At the corner $t \downarrow 0$, $x \to \infty$, use a compact band strictly above $y = 1$ instead. On slightly larger disks choose a fixed $\sigma_* > 1$ below all real parts of w. Combining the finite heat exponents gives

$$|P_N(s_B,t) - \zeta(s_B)| \leq \frac{t}{4}\sum_{n\geq 1}\log^2 n\, n^{-\sigma_*} + \sum_{n>N} n^{-\sigma_*}. \tag{A.13}$$

If necessary choose two fixed exponents between 1 and the smallest $\Re w$ to absorb the vanishing archimedean errors. The reflected term and the remainder tend uniformly to zero. Both sides of the resulting difference $H_t/B_t - \zeta(s_B) = O(t) + o(1)$ are holomorphic, so Cauchy twice gives the same conclusion for their derivatives. The Euler product bounds $\zeta(s_B)$ uniformly away from zero. Since $s_{B,z} = -i/2 + o(1)$ and $s_{B,zz} = o(1)$, it follows that

$$S = \tfrac{1}{4}\log(x/(4\pi)) + O(1), \qquad \limsup_{t\downarrow 0,\ x\to\infty} J(x,y,t) \leq \frac{\zeta''((1+y)/2)}{4\zeta((1+y)/2)}. \tag{A.14}$$

For the sharp constant in (A.14), put $q = \zeta'/\zeta$ and $W(\sigma) = -q(\sigma)$. The Euler logarithm gives $|q(s)| \le W(\sigma)$ and $|q'(s)| \le q'(\sigma)$ when $\Re s \ge \sigma > 1$; hence $(|q'(s)| + |q(s)|^2)/4 \le \zeta''(\sigma)/(4\zeta(\sigma))$. Use $\Re s_B \ge (1+y)/2 - o(1)$ and continuity to obtain the displayed limsup. It is uniform on the compact band. The argument allows $t \log x$ to be unbounded; it does not replace $\zeta(s_B)$ by $\zeta(w)$. For zero contacts themselves, independently of (A.12), [Po19, Theorem 1.5(i)] supplies a uniform finite horizontal confinement on each compact positive-time interval. An initial strict strip cushion will keep this theorem from being used at $t = 0$.

A.4. The coupled density and strip comparison

Let q_M be the continuous affine function defined by the 1620 exact rows of `certificate/data/old-M3a-wall.json`. Its initial value is h_0^2, $h_0 = 1000001/10^6$, its terminal value is $1/400$, and its horizon is

$$T_M = \frac{107240232634533137129654154921967198029 51}{662072111959368380015602207712500000000 0}.$$

Write $h_M(t) = \sqrt{q_M(t)}$, and define

$$\begin{aligned} U(Y,t) &= 10^9 + \sum_{j=1}^{3} a_j e^{\lambda_j^2 t} \cosh(\lambda_j Y), \\ (a_1, a_2, a_3) &= (1758974, 2464729, 302096), \qquad (\lambda_1, \lambda_2, \lambda_3) = (19/4, 97/20, 131/20), \\ f_0 &= U_Y/U, \qquad \delta(t) = e^{90t}/10^{14}, \qquad f = f_0 - \delta. \end{aligned} \tag{A.15}$$

Theorem A.5 (density and intermediate strip). For every $0 \le t \le T_M$, all zeros of H_t lie in $|\Im z| \le h_M(t)$. For every $x \ge X$ and $h_M(t) < Y \le 5$, $H_t(x+iY) \ne 0$ and

$$S(x, Y, t) \ge f(Y, t). \tag{A.16}$$

We prove these two assertions simultaneously. The finite-head theorem P2, the initial critical strip, and the classical zero dynamics of the main text are inputs. No later jet estimate or later reference barrier is used.

First, $U_t = U_{YY}$, so $f_{0,t} = f_{0,YY} + 2f_0 f_{0,Y}$. Expanding the coshes into positive exponentials makes $f_{0,Y}$ the variance of the slopes $0, \pm\lambda_1, \pm\lambda_2, \pm\lambda_3$ under their normalized positive weights. Consequently

$$0 < f_{0,Y} \le \lambda_{\max}^2 = 17161/400, \quad 0 < f_0 < \lambda_{\max} = 131/20 \quad (Y > 0),$$

and

$$f_{YY} + 2f f_Y - f_t = (90 - 2f_{0,Y})\delta \ge (839/200)\delta > 0. \tag{A.17}$$

At fixed $Y > 0$, group the positive cosh weights by their nonnegative frequency, including a constant mode at frequency zero. Then f_0 is the weighted mean of $m(\lambda) = \lambda \tanh(\lambda Y)$ and

$$\partial_t f_0 = \frac{1}{2} \sum_{j,k} w_j w_k [m(\lambda_j) - m(\lambda_k)](\lambda_j^2 - \lambda_k^2) \ge 0. \tag{A.18}$$

Both functions in the covariance increase with $\lambda \geq 0$. It is f_0, rather than the subtracted function f, whose time monotonicity is used in all row bounds. The scalar check gives $\delta(T_M) < 2.144 \cdot 10^{-8} < 10^{-5}$.

For the initial density, (A.6), Cauchy on a unit w-disk, and the absolutely convergent Euler series give, for $x \geq X_L$, $1 < Y \leq 5$,

$$S(x, Y, 0) \geq \frac{1}{4} \log \frac{x}{4\pi} + \frac{\zeta'((1+Y)/2)}{2\zeta((1+Y)/2)} - \frac{1}{x^2} - \frac{1}{6x - 36} > E(Y), \tag{A.19}$$

where $E(Y) = \frac{1}{4} \log(X_L/(4\pi)) - 10^{-6} + \zeta'((1+Y)/2)/(2\zeta((1+Y)/2))$. The last two errors in (A.19) are less than $2.779 \cdot 10^{-14}$. The function E increases, since its derivative is $\frac{1}{4} \sum_{n \geq 1} \Lambda(n) \log n \, n^{-(1+Y)/2} > 0$.

An additional kernel comparison is useful close to $Y = 1$. For $1 < u < v$ and $0 \leq b \leq 1$, clearing denominators in $K_u(d, b)/u - K_v(d, b)/v$ leaves a positive factor times

$$(v^2 - u^2)\{D^2 + (u^2 + v^2 - 6b^2)D + (u^2 - b^2)(v^2 - b^2)\}, \quad D = d^2.$$

The expression in braces decreases with $b^2 \in [0, 1]$. At $b = 1$ it is nonnegative for all $D \geq 0$ precisely when

$$u^2 + v^2 - 6 \geq 0 \quad \text{or} \quad 4(u^2 - 1)(v^2 - 1) \geq (u^2 + v^2 - 6)^2. \tag{A.20}$$

If (A.20) holds and $E(v) > 0$, summing the absolutely convergent paired zero kernels gives $S(x, u, 0) > (u/v)E(v)$. Admissibility persists as u increases toward v, because the brace has positive derivative with respect to u^2.

`check_m3a.py` reconstructs the 855 dyadic leaves of $[h_0, 5]$ in `support/fields/data/m3a-initial.json`, and checks their exact adjacency. On 851 leaves $[l, r]$ it verifies $E(l) - f_0(r, 0) > 10^{-7}$. On the other four, it verifies (A.20) at $u = l$, $v \geq r$, $E(v) > 0$, and $(l/v)E(v) - f_0(r, 0) > 10^{-7}$. Their reference heights are 447/200, 179/100, 1601/1000, and 193/125. Monotonicity then proves the strict initial bound on every point of every leaf. The smallest computed gap is greater than $4.7661468 \cdot 10^{-7}$.

The moving bottom has a separate, exact nonlocal inequality. Put $A_j = a_j e^{\lambda_j^2 t}$, $r_j = \sinh(\lambda_j h)/(\lambda_j h)$ and

$$V = \sum_j A_j \cosh(\lambda_j h), \quad W = \sum_j A_j \lambda_j^2 r_j, \quad D = \sum_j A_j \lambda_j^4 r_j^3, \quad E_* = \sum_j A_j \lambda_j^2 \cosh(\lambda_j h) r_j^2.$$

Triple-angle identities give

$$\frac{f_0(3h, t)}{3} - f_0(h, t) = \frac{4h^3[D(10^9 + V) - 3WE_*]}{3U(3h, t)U(h, t)}. \tag{A.21}$$

Each of D, V, W, E_* increases with both t and $h > 0$. For r_j this follows from $v \cosh v - \sinh v > 0$, whose derivative is $v \sinh v$. On a decreasing row $[t_l, t_r]$ with endpoint squares $q_l > q_r$, a lower bound for the numerator factor in (A.21) is therefore

$$D(t_l, \sqrt{q_r})[10^9 + V(t_l, \sqrt{q_r})] - 3W(t_r, \sqrt{q_l})E_*(t_r, \sqrt{q_l}). \tag{A.22}$$

The checker verifies positivity on all 1620 rows, with minimum greater than $3.4851 \cdot 10^{19}$, and verifies $3h_M \le 5$. This rectangle estimate does not require the difference in (A.21) itself to be monotone. Subtracting δ adds $2\delta/3$ to (A.21), giving $f(3h_M, t)/3 > f(h_M, t)$.

The spatial boundary is supplied by the finite-head theorem. If $|x| \le X - 1$ and $Y > 0$, every nonreal zero has horizontal distance at least 1 from x. Its paired kernel is

$$K_Y(d, b) = \frac{2Y(d^2 + Y^2 - b^2)}{[d^2 + (Y - b)^2][d^2 + (Y + b)^2]} > 0$$

because $|b| \le 1$. Real-root contributions are positive as well. Thus $p = iH'/H$ is holomorphic with positive real part in the radius-five disk centered at $x + 5i$ when $X_e + 6 \le x \le X - 6$. The Cayley transform and Schwarz's lemma give

$$|\Re u(x + iY)| \le |\Re u(x + 5i)| + (10/Y - 1)S(x, 5, t) < 2000 \quad (1/20 \le Y \le 5), \tag{A.23}$$

using Lemma A.4. We use the looser constant $M = 20000$. Choose a smooth nondecreasing χ, zero on $(-\infty, 0]$ and one on $[1, \infty)$, and set

$$g(x, t) = 10e^{86t}\chi\Big(\frac{X - 100000 + 40000t - x}{900000}\Big), \qquad F = f - g. \tag{A.24}$$

Then $g = 0$ for $x \ge X$ and $g \ge 10$ at $x = X_L$. The transition lies in $[X_L, X - 92000] \subset [X_e + 6, X - 6]$ for $t \le 1/5$. There $g_x \le 0$ and $g_t = 86g + 40000|g_x|$.

On a zero-free region write $u = A - iS$. The heat equation and the Cauchy–Riemann equations give the exact vertical equation

$$S_t = S_{YY} - 2AS_x + 2SS_Y. \tag{A.25}$$

Equations (A.17), (A.23), and $86 - 2\lambda_{\max}^2 = 39/200$ imply

$$F_{YY} - 2AF_x + 2FF_Y - F_t \ge (839/200)\delta + (39/200)g + (40000 - 2M)|g_x| > 0. \tag{A.26}$$

Where $g_x = 0$ no bound on A is needed. At the left boundary $F < 0 < S$; at the top $F < f_0 < 6.55 < 6.59 < S$. The initial comparison follows from (A.19)–(A.20). At the bottom, $F(3h_M, t)/3 - F(h_M, t) > 0$ by (A.21) and $2g/3 \ge 0$.

Suppose a first failure of the strip or the density bound occurs. Initially $h_0 > 1$, and by continuity $h_M > 1 + \eta$ for some positive initial time interval. The first formula in (A.14) excludes density contacts escaping to infinity on that interval; its compact complement is handled by the strict initial margin. Equation (A.12) excludes later density escape. The initial strip cushion and [Po19, Theorem 1.5(i)] exclude height contacts escaping to infinity. Thus any first contact occurs at a finite point and a positive time.

Until that time all roots stay in the closed h_M strip. For $h_M < Y \le 3h_M/\sqrt{5}$, the unsigned kernel inequality in the main text gives $S(x, Y, t) \ge (Y/(3h_M))S(x, 3h_M, t)$. Taking a lower limit, also as x, t vary, gives

$$\liminf_{Y \downarrow h_M}(S - F) \ge F(x, 3h_M, t)/3 - F(x, h_M, t) > 0. \tag{A.27}$$

The reference $3h_M$ stays outside the root strip. Thus (A.27) excludes bottom contacts even at a possible pole, without asserting an infinite liminf along every approach to that pole. At an interior density contact, $S_x = F_x$, $S_Y = F_Y$, and $S_{YY} \geq F_{YY}$. Equations (A.25)–(A.26) give $(S-F)_t > 0$, contradicting first contact from positive values. No horizontal diffusion term is assumed.

A first height contact is outside $|x| \leq X$ by P2; symmetry puts it at $x > X$, where $g = 0$. For every exact row the checker verifies

$$\begin{gathered} 0 < t_r - t_l \leq 10^{-4}, \quad 1/400 \leq q_r < q_l, \quad 0 < p \leq 5, \quad p^2 \geq 5q_l, \\ s < f_0(p, t_l) - 10^{-5}, \qquad k = 4s/p - 8/(p^2 - q_l) > 0, \qquad \frac{q_l - q_r}{t_r - t_l} \leq 2 + kq_r. \end{gathered} \tag{A.28}$$

By (A.18), $S(x,p,t) \geq f_0(p,t) - \delta(t) > s$ throughout the closed row. At a simple highest zero the unsigned force law in the main text therefore gives $(h^2)' < -2 - kq_M(t) \leq -2 - kq_r \leq q_M'$. This contradicts first contact. A multiple contact is excluded by backward Hermite splitting: an earlier zero has imaginary part greater by a positive multiple of $\sqrt{\varepsilon}$, while h_M changes by $O(\varepsilon)$. At a time knot use the preceding row and the left derivative. Simultaneous contacts need only the non-strict other bound at the first time, so both strict contradictions remain valid. In fact the argument proves $S(x,Y,t) \geq f(Y,t) - g(x,t)$ on the larger domain $x \geq X_L$, $h_M(t) < Y \leq 5$, at every time in $[0, T_M]$. Its restriction to $x \geq X$, where $g = 0$, is Theorem A.5. This larger intermediate density field is the spatial estimate used in A.5.

A.5. The fixed jet supersolution

Define

$$\Omega(x) = \frac{1}{4}\log\frac{x}{4\pi}, \quad c = -\pi/8, \quad J = |u_z| + |u - c + i\Omega|^2, \quad b(t) = h_M(t) + 3/5,$$

and

$$C(Y,t) = e^{\kappa t}\left[\frac{8}{25} + \frac{A_*}{U(Y,t)}\right], \qquad \kappa = 10^{-5}, \quad A_* = 1130000000000. \tag{A.29}$$

Theorem A.6 (jet bound). For every $x \geq X$, $0 \leq t \leq T_M$, and $b(t) \leq Y \leq 5$,

$$J(x,Y,t) \leq C(Y,t). \tag{A.30}$$

The band is separated by 3/5 from every zero, by Theorem A.5. Its lower-boundary input is the estimate

$$J(x, b(t), t) < C(b(t), t) \quad (x \geq X_L, 0 \leq t \leq T_M), \tag{A.31}$$

which is proved directly in A.6–A.7 below. We first prove that this input implies (A.30), keeping every other boundary and the unbounded-domain issue.

Put $u = A - iS$, $V = -u_z = R + iP$, and $\mathcal{L} = \partial_t - \partial_{YY} + 2A\partial_x - 2S\partial_Y$. Direct differentiation of the heat equation gives $\mathcal{L}u = 0$, $\mathcal{L}V = 2V^2$. With $a = A - c$, $d = S - \Omega$, define the smooth symmetric matrix

$$\mathcal{M} = (a^2 + d^2)I + \begin{pmatrix} R & P \\ P & -R \end{pmatrix}.$$

Its largest eigenvalue is J. The product rule, including $\Omega' = 1/(4x)$, gives

$$\mathcal{L}(a^2+d^2) = -2(R^2+P^2) - Ad/x, \qquad \mathcal{L}\mathcal{M} = -4\begin{pmatrix} P \\ -R \end{pmatrix}\begin{pmatrix} P & -R \end{pmatrix} - (Ad/x)I. \tag{A.32}$$

At a prospective upper contact $J = D > 0$, take a unit eigenvector at that point and then hold it constant locally. The smooth scalar $\varphi = v^T \mathcal{M} v$ is everywhere at most J, equals J at contact, and satisfies

$$\mathcal{L}\varphi \le -Ad/x \le E(D) := \frac{\sqrt{D}(\pi/8+\sqrt{D})}{X_L}. \tag{A.33}$$

The inequalities $|A-c| \le \sqrt{D}$ and $|d| \le \sqrt{D}$ justify the last step. This argument also covers $u_z = 0$ and a repeated eigenvalue. It does not differentiate an eigenvector or the nonsmooth function $|u_z|$.

Since A_*/U satisfies $v_t = v_{YY} + 2f_0 v_Y$, (A.29) gives

$$C_t - C_{YY} - 2fC_Y = \kappa C + 2\delta C_Y, \qquad C_Y \le 0, \quad |C_Y| \le \lambda_{\max} C, \quad C \ge 8/25.$$

Thus its relative supersolution margin over (A.33) is at least

$$\kappa - 2\lambda_{\max}\delta(T_M) - \frac{1+\pi/(8\sqrt{8/25})}{X_L} > 9.7192 \cdot 10^{-6}. \tag{A.34}$$

The exact source term in (A.32) is retained; the center $c - i\Omega(x)$ is not a constant holomorphic function.

At $t = 0$ and $1 < Y \le 7$, write $G = \zeta'/\zeta(w)$. Use (A.6) on a w-disk of radius $x/4$; its gamma argument has imaginary part at least $x/8$. Define

$$R_\Gamma(x) = \frac{1}{12(x/8-1)}, \quad e_u = \frac{4}{x-20} + \frac{2R_\Gamma(x)}{x}, \quad e_V = \frac{1/x+6/x^2}{4} + \frac{8R_\Gamma(x)}{x^2}.$$

Equations (A.2), (A.6), and Cauchy give $|u-(c-i\Omega)| \le |G|/2 + e_u$ and $|u_z| \le |G'|/4 + e_V$. For real $\sigma = (1+Y)/2 > 1$, put $W = -\zeta'(\sigma)/\zeta(\sigma)$. The absolutely convergent Euler logarithm yields

$$J(x,Y,0) \le C_{\text{in}}(Y) := \frac{\zeta''(\sigma)}{4\zeta(\sigma)} + e_V + We_u + e_u^2, \tag{A.35}$$

using the decreasing error terms at $x = X_L$. Indeed $|G| \le W$, $|G'| \le (\log\zeta)''(\sigma)$ and $(\log\zeta)'' + W^2 = \zeta''/\zeta$. Both W and ζ''/ζ decrease with σ; for the latter, differentiate the expectation of $\log^2 n$ under weights $n^{-\sigma}/\zeta(\sigma)$ to get $-\operatorname{Cov}(\log^2 n, \log n) \le 0$.

The fixed-C checker divides $[b(0), 3]$ into 400 equal exact rational cells $[l, r]$ and verifies $[C_{\text{in}}(l) - 8/25]_+ U(r, 0) < A_*$ on each. The maximum is less than $1128379860282.031 < A_*$. For $3 \le Y \le 5$, $C_{\text{in}}(Y) \le C_{\text{in}}(3) < 0.302335 < 8/25$. This proves a uniform strict initial margin on the entire required band. Only the finite range $1 < Y \le 7$ is asserted in (A.35). Lemma A.4 gives the strict top margin.

For the left boundary, Schwarz–Pick in the same positive-real-part disk used for (A.23) gives

$$|u(x+iY)| < 100/Y, \qquad |u_z(x+iY)| \le \frac{10S(x,Y,t)}{Y(10-Y)} < 100/Y^2.$$

Since $Y \ge b(t) \ge 13/20$ and $|c - i\Omega(X_L)| < 7$, $J(X_L, Y, t) < 4412281/169 < 30000$. Let $G_* = 10^6 g$, with g from (A.24), and use the full spatial supersolution $D = C + G_*$. Then $D = C$ for $x \ge X$ and $G_*(X_L, t) \ge 10^7$. Throughout the transition $\mathcal{L}G_* \ge 86 \cdot 10^6 g$. The actual density estimate there is $S \ge f - g$, so

$$\mathcal{L}C \ge C_t - C_{YY} - 2fC_Y - 2g|C_Y|.$$

The checker bounds $C < 103.119 < 10^6$ on the entire band, using the opposite time/height corners of every q_M row. Also $E(C+G_*) - E(C) \le L_E G_*$, where $L_E = [1 + \pi/(16\sqrt{8/25})]/X_L$. The strict inequality $(86 - L_E)10^6 > 2\lambda_{\max}10^6$ therefore proves $\mathcal{L}D > E(D)$, with the positive margin from (A.34). All spatial derivative terms are included. Outside the transition $g_x = 0$, so no bound on $\Re u$ is needed there.

Finally, (A.14) and the strict initial margin (A.35) exclude a first upper contact escaping through $t = 0, x = \infty$. By continuity the margin persists on a small neighborhood below $b(0) > 1$, which contains the short-time moving band. Compact x is handled by ordinary joint continuity. For times bounded away from zero, (A.12) and $D \ge 8/25$ exclude escape to infinity. Hence a first contact, if present, is finite and at positive time. Initial, left, top, and bottom contacts are excluded by the strict bounds already established, including (A.31). At an interior contact the fixed-eigenvector scalar from (A.32) satisfies $\varphi - D \le 0$, equals zero, has vanishing spatial first derivatives, and has nonpositive Y second derivative. First contact from below gives its time derivative a nonnegative left limit. Thus $\mathcal{L}(\varphi - D) \ge 0$, contrary to (A.33) and $\mathcal{L}D > E(D)$. At a knot of b use the preceding closed interval; C itself is smooth. This proves Theorem A.6 once (A.31) is supplied.

A.6. The early boundary: a relative Euler expansion

The 162 exact boundary rectangles are defined by $t_i = i/1000$ for $0 \le i \le 161$, $t_{162} = T_M$, and

$$y_i^- = \frac{3}{5} + 10^{-6}\lfloor 10^6 h_M(t_{i+1})\rfloor, \qquad y_i^+ = \frac{3}{5} + 10^{-6}\lceil 10^6 h_M(t_i)\rceil. \tag{A.36}$$

Then $b(t) \in [y_i^-, y_i^+]$ on $[t_i, t_{i+1}]$. The fixed-C checker reconstructs these square inequalities, their rounding intervals, every q_M interpolation, and the exact time cover from the local geometry file. It also checks that all physical radius-1/4 disks and source radius-3/20 disks stay in $x > X_e$, $0 < y < 5/2$, with cutoff at least N_0.

For boxes $0, \dots, 44$, the center heights lie in $[1353931/10^6, 1600001/10^6]$, so every radius-3/20 source disk centered in the center rectangle stays above $y = 1$. This direct method does not use source disks centered on an additional radius-1/4 circle. The resulting closed time interval includes $t = 0$ and $t = 9/200$. For one box $[t_l, t_r] \times [y_l, y_r]$, put

$$\begin{aligned}
&\ell_0 = \log N_0, && e_0 = \log(1 - t_m/(16N_0^2)), \\
&c_* = -t_m e_0/4, && d_* = t_m/X_e^2, \\
&c_y = (1+y)/2 - c_* - d_*, && c_0 = c_{y_l}, \qquad a = c_0 + t_l\ell_0/2.
\end{aligned}$$

The cutoff relation and (A.2) give

$$\Re s_B \geq c_y + (t/2)\log N \geq a > 1. \tag{A.37}$$

The small errors remain inside exponents when $\log N$ is unbounded. Let

$$\begin{aligned}
A_0 &= \tfrac{1}{2}\sqrt{\log^2(X_e/(4\pi)) + \pi^2/4} + 8/(X_e - 20),\\
A_1 &= 1/(X_e - 6), \quad A_2 = 2/X_e^2 + 24/X_e^3,\\
r_1 &= 500001/10^6, \quad r_2 = t_m A_2/8,\\
D_\beta &= 8/(X_e - 20) + (t_m/2)A_0A_1,\\
B_\beta &= A_1 + (t_m/2)(A_1^2 + A_0A_2), \quad \epsilon_1 = D_\beta/2, \quad \epsilon_2 = B_\beta/4.
\end{aligned}$$

These bound $|s_{B,z}|$, $|s_{B,zz}|$, $|B'/B+\pi/8+i\Omega|$, and $|(\log B)''|$ by $r_1, r_2, \epsilon_1, \epsilon_2$, respectively. Every product containing $A_0(x)$ decreases after combination with its inverse powers: $xZ'(x) \leq 1$, $Z(x) > 1$, so $Z(x)/(x-d)^k$ decreases for $d = 6, 20, 40$ and $k \geq 1$. The same argument controls $\frac{1}{2}\log(x/(4\pi) + t_m/16)$ times these inverse powers. The growing logarithm by itself is never bounded at X_e.

For a locally fixed actual cutoff, the identity

$$\begin{aligned}
P_N(s_B, t) &= \zeta(s_B) + (t/4)\zeta''(s_B) + R_H - T_N,\\
R_H &= \sum_{n\leq N}[e^{t\log^2 n/4} - 1 - t\log^2 n/4]n^{-s_B},\\
T_N &= \sum_{n>N}[1 + (t/4)\log^2 n]n^{-s_B}
\end{aligned} \tag{A.38}$$

is exact. The heated sums are finite; the infinite sum is ordinary and absolutely convergent. Write

$$H_t/B_t = \zeta(s_B)(1+\rho), \qquad \rho = \frac{t}{4}\frac{\zeta''}{\zeta}(s_B) + \frac{R_H - T_N + \gamma P_N(s_A, t) + E_N}{\zeta(s_B)}, \tag{A.39}$$

where $E_N = H_t/B_t - f^{[N]}$ is holomorphic for this fixed cutoff. We now bound the three raw jets of the remainder numerator.

For $j = 0, 1, 2$, let H_j bound the positive sum

$$\sum_{n\geq 2}\log^j n\, n^{-c_0}\exp\left[t_l\left(\frac{\log^2 n}{4} - \frac{\log n\,\max(\ell_0, \log n)}{2}\right)\right]h(t_r\log^2 n/4), \quad h(v) = 1-(1+v)e^{-v}. \tag{A.40}$$

Here $h \geq 0$ increases and $h'/h \leq 2/v$, since $h(v) = \int_0^v re^{-r}\,dr \geq v^2e^{-v}/2$. For $n \leq N$, combine the heat factors with (A.37) and use $\log N \geq \max(\ell_0, \log n)$ before extending the majorant to all n. The combined heat exponent is nonpositive, so its time is lowered to t_l; the increasing h is bounded at t_r. The logarithmic derivative of the resulting summand is at most $-c_0+(j+4)/\log n$. The gate $c_0 > 6/\log 1024$ makes it decreasing beyond 1024.

The checker sums $2 \leq n \leq 1024$ and bounds the rest by an integral. In log coordinates, $\mathcal{E}(v) = \exp[-(c_0-1)v + t_l(v^2/4 - v\max(\ell_0, v)/2)]$ decreases, while $v^jh(t_rv^2/4)$ increases. On 1024 intervals with exact symbolic endpoints $v_k = \log 1024 + k/16$ use their left exponential and right polynomial/h factors. The remaining tail from $V = \log 1024 + 64 > \ell_0$ is at most

$$\mathcal{E}(V)\sum_{k=0}^{j}\binom{j}{k}\frac{V^{j-k}k!}{(c_0-1+t_lV/2)^{k+1}}. \tag{A.41}$$

This follows from $\mathcal{E}(V+w)\le\mathcal{E}(V)e^{-(c_0-1+t_lV/2)w}$ and $h\le 1$. Every rate is checked positive, including the $t_l=0$ box.

The ordinary tails use

$$I_j=N_0^{1-a}\sum_{k=0}^{j}\frac{j!}{(j-k)!}\frac{\ell_0^{j-k}}{(a-1)^{k+1}},\qquad T_j=I_j+(t_r/4)I_{j+2}\quad(j=0,1,2). \tag{A.42}$$

Here $I_j=\int_{N_0}^{\infty}u^{-a}\log^j u\,du$. The gate $a>4/\ell_0$ proves the decreasing sum-integral comparison for every moment through degree four and every actual $N\ge N_0$.

For the reflected term put

$$K=\frac{t_m}{2(4\pi N_0^2-\pi t_m/4-6)},\quad \Delta=10^{-11}-e_0/2,\quad b=c_0-y_l-Ky_l,\quad e=b+y_l,\quad a_0=1-b.$$

After combining gamma and polynomial factors, the coefficient of y is $\Delta-\log N+(1/2+K)\log n<0$, by a gate at N_0. Its maximum therefore occurs at y_l. For $L_N=\log N$, the remaining summand $g(u)=\exp[-(b+t_lL_N/2)\log u+t_l\log^2 u/4]$ has a single valley. Thus $\sum_{n\le N}g(n)\le 1+\int_1^N g(u)\,du+g(N)$. After multiplication by $e^{-y_lL_N}$, its integral is

$$\mathcal{A}(L_N)=e^{-y_lL_N}\int_0^{L_N}e^{(a_0-t_lL_N/2)v+t_lv^2/4}\,dv, \tag{A.43}$$

and its endpoints are $e^{-y_lL_N}$ and $e^{-eL_N-t_lL_N^2/4}$. The first and second raw phase multipliers are bounded by $L_N+\beta_0$ and τ_2, where $\beta_0=1/1000$, $\tau_2=10^{-10}$. Differentiate $\log\gamma-s_A\log n$ directly; the leading opposite imaginary phases in $\alpha(v)+\alpha(w)$ cancel. The residual bounds are

$$1/N_0+D_\beta+\frac{t_m\ell_0}{4(X_e-6)}<\beta_0,\qquad B_\beta/2+r_2\ell_0<\tau_2. \tag{A.44}$$

The inverse-power/logarithm monotonicities already stated extend these to every x,N. This calculation is for the raw reflected sum.

The square phase weight also has an all-N reduction. Set $K_0=(1-e^{-a_0\ell_0})/a_0$. The checker verifies

$$a_0>0,\qquad (y_l-2/(\ell_0+\beta_0))K_0>1,\qquad e(\ell_0+\beta_0)>2. \tag{A.45}$$

If $\mathcal{B}$ is the upper-endpoint integrand in (A.43), then $\mathcal{A}'\le -y_l\mathcal{A}+\mathcal{B}$ and $\mathcal{A}/\mathcal{B}\ge\int_0^{L_N}e^{-a_0w}\,dw\ge K_0$. The latter follows from the exact exponent difference $-a_0w+t_lw^2/4$ at $v=L_N-w$. Thus $(L_N+\beta_0)^2\mathcal{A}(L_N)$ decreases for $L_N\ge\ell_0$. The two endpoint terms with this weight decrease by (A.45) too. At $L_N=\ell_0$, the integrand increases because $1-b-t_l\ell_0/2>0$, another checked gate. Use 1024 right-endpoint rectangles there. If A_R is this full reflected magnitude bound, including both endpoints and $e^{\Delta y_l}$, its first two derivatives are bounded by $(\ell_0+\beta_0)A_R$ and $[(\ell_0+\beta_0)^2+\tau_2]A_R$.

Lemma A.3 on the full radius $r = 3/20$ disk gives E_0, E_1, E_2, where E_0 is the point error, $E_1 = E_D/r$ and $E_2 = 2E_D/r^2$. The second derivative factorial is retained. Positive source factors may use N_e in place of N_0; the cutoff corrections themselves use N_0. The raw remainder jets in (A.39) are therefore bounded by

$$\begin{aligned} R_0 &= H_0 + T_0 + A_R + E_0, \\ R_1 &= r_1(H_1 + T_1) + (\ell_0 + \beta_0)A_R + E_1, \\ R_2 &= r_1^2(H_2 + T_2) + r_2(H_1 + T_1) + [(\ell_0 + \beta_0)^2 + \tau_2]A_R + E_2. \end{aligned} \tag{A.46}$$

Here H_j is the moment (A.40), not the heat-flow function. For $W_j(a) = \sum_{n\geq 1} \Lambda(n) \log^j n\, n^{-a}$, put

$$Z_0 = W_1 + W_0^2, \quad Z_1 = W_2 + 2W_0W_1, \quad Z_2 = W_3 + 2W_1^2 + 2W_0W_2, \quad m_\zeta = \zeta(2a)/\zeta(a).$$

The Euler product gives $|\zeta(s)| \geq m_\zeta$ for $\Re s \geq a$; the Z_j bound the first three s-jets of ζ''/ζ. Product and quotient differentiation in (A.39) give

$$\begin{aligned} p_0 &= t_r Z_0/4 + R_0/m_\zeta, \\ p_1 &= t_r r_1 Z_1/4 + (R_1 + r_1 W_0 R_0)/m_\zeta, \\ p_2 &= t_r(r_1^2 Z_2 + r_2 Z_1)/4 \\ &\qquad + [R_2 + 2r_1 W_0 R_1 + \{r_1^2(W_0^2 + W_1) + r_2 W_0\}R_0]/m_\zeta. \end{aligned} \tag{A.47}$$

Thus $|\rho^{(j)}| \leq p_j$. With $d = 1 - p_0 > 0$, put $g_1 = r_1W_0 + p_1/d$ and $g_2 = r_1^2W_1 + r_2W_0 + p_2/d + (p_1/d)^2$. Then $H_t \neq 0$ and

$$J \leq \epsilon_2 + g_2 + (\epsilon_1 + g_1)^2. \tag{A.48}$$

The checker computes the W_j from the Taylor series of $\log\zeta(a+u)$ with the exact coefficient factorials. For every box it verifies $d > 0$ and compares (A.48) with

$$C_i^- = e^{t_i/100000}\left[8/25 + A_*/U(y_i^+, t_{i+1})\right]. \tag{A.49}$$

Monotonicity of U in time and positive height gives $C(Y,t) \geq C_i^-$ on the whole rectangle. The program check_early_boundary.py verifies all 45 strict gates, with minimum margin greater than 17.07745. This proves (A.31) on $[0, 9/200]$ for every $x \geq X_L$ and actual cutoff.

A.7. The late boundary: mollified derivative bounds

For boxes $45, \ldots, 161$ define the cutoff-independent holomorphic function

$$\mathcal{E}_t(s) = \prod_{p\in\{2,3,5,7,11\}} (1 - b_t(p)p^{-s}), \qquad \mathcal{F}(z) = \mathcal{E}_t(s_B(z))H_t(z)/B_t(z). \tag{A.50}$$

For each time interval evaluate the center-height rectangle and its enlargement by $1/4$ on either side. Their horizontal lower bounds are X_L and $X_L - 1/4$. Every source disk has radius $3/20$. $\mathcal{F}$ and $\mathcal{F}'$ are holomorphic on these disks even where an estimate of $|\mathcal{F}|$ from below is unavailable.

The actual cutoffs are partitioned into eight finite bands $[n, m]$, starting at $n = N_0$, with $m = \lfloor 11n/10 \rfloor$ and then $n \leftarrow m + 1$, followed by $[1481120, \infty)$. They are exact adjacent integer intervals. For a band starting at n, $x \geq x_b = \max(x_{\min}, 4\pi n^2 - \pi t_m/4)$. Write $y = y_l$, $t_0 = t_l$, $L_n = \log n$, and put

$$\begin{aligned} &c_A = (1 + y)/2 - c_* - d_*, \quad c_B = c_A - y - Ky, \quad c_E = c_B + y, \\ &\sigma_0 = c_A + t_0 L_n/2, \quad \Delta = 1/50 - e_0/2, \quad \beta_1 = 1. \end{aligned} \tag{A.51}$$

The constants e_0, c_*, d_*, K are those of A.6. Equations (A.2)–(A.3), with the permissible larger gamma allowance $1/50$, imply

$$\Re s_B \geq c_A(Y) + (t/2)\log N, \quad \Re s_A \geq c_B(Y) + (t/2)\log N, \quad |\gamma| \leq e^{\Delta Y} N^{-Y}. \tag{A.52}$$

The formulas below specify the all-cutoff calculation in late_core.py. Let $\mathcal{D}$ be the 32 squarefree divisors of 2310 and $\ell_d^{(2)} = \sum_{p|d} \log^2 p$. With $R = 100000 < N_0$, the exact coefficient of r^{-s_B} in the mollified first polynomial for $r \leq R$ is

$$B_r(t) = \sum_{\substack{d \in \mathcal{D} \\ d|r}} \mu(d) \exp\left[\frac{t}{4}\{\ell_d^{(2)} + \log^2(r/d)\}\right]. \tag{A.53}$$

All these terms occur for every actual N and $B_1 = 1$. The program encloses this signed sum on the whole time interval before taking absolute values. Define

$$S_B = \sum_{r=2}^{R} |B_r| r^{-\sigma_0}, \qquad D_B = \sum_{r=2}^{R} \log r\, |B_r| r^{-\sigma_0}. \tag{A.54}$$

Every omitted divisor term has $u > \lfloor R/d \rfloor$ and $u \leq N$. For a finite band take $M = m$, $H = \log M$; for the unbounded band use $M = n$, $H = L_n$ on its finite prefix. The program checks, for every divisor,

$$H < 2L_n, \quad c_A - \frac{t_0}{2}(H - L_n) > \frac{1}{\log d + \log\lfloor R/d \rfloor}, \quad \ell_d^{(2)} - 2L_n \log d \leq 0. \tag{A.55}$$

The combined heat exponent is nonincreasing in time, so the left endpoint is valid. The middle gate makes both the summand and its $\log(du)$-weighted version decreasing on the finite interval. Put $I_j(a, t; l, M) = \int_l^M u^{-a} e^{t \log^2 u/4} \log^j u\, du$ for $j = 0, 1$. The finite tails are bounded by

$$\begin{aligned} T_B &= \sum_{d \in \mathcal{D}} e^{t_0 \ell_d^{(2)}/4 - \sigma_0 \log d} I_0(\sigma_0, t_0; \lfloor R/d \rfloor, M), \\ D_T &= \sum_{d \in \mathcal{D}} e^{t_0 \ell_d^{(2)}/4 - \sigma_0 \log d} [\log d\, I_0 + I_1](\sigma_0, t_0; \lfloor R/d \rfloor, M). \end{aligned} \tag{A.56}$$

The upper endpoint is M, rather than M/d, retaining every original polynomial term after multiplication by a divisor. On the unbounded band replace $\log N$ by $\max(L_n, \log u)$ only after combining its nonpositive coefficient. For $u > n$ this gives a negative Gaussian in $\log u$. In (A.56) add, to each I_j,

$$J_j(c_A, t_0; n) = \int_n^\infty u^{-c_A} e^{-t_0 \log^2 u/4} \log^j u\, du. \tag{A.57}$$

The gates $a+t_0L_n/2>1/L_n$ for both $a=c_A$ and $a=c_E$ prove the decreasing tail comparisons here and for the reflected tail below. Thus all $N\ge n$ are covered. All integrals are computed by exact antiderivatives and ball arithmetic. For $k=1-a$, $l_0=\log l$, $h_0=\log M$, and $t>0$,

$$\begin{aligned} I_0 &= \sqrt{\pi/t}\,e^{-k^2/t}[\operatorname{erfi}(\sqrt{t}\,h_0/2+k/\sqrt{t})-\operatorname{erfi}(\sqrt{t}\,l_0/2+k/\sqrt{t})],\\ I_1 &= \frac{2}{t}[e^{th_0^2/4+kh_0}-e^{tl_0^2/4+kl_0}-kI_0],\\ J_0 &= \sqrt{\pi/t}\,e^{k^2/t}\operatorname{erfc}(\sqrt{t}\,L_n/2-k/\sqrt{t}),\\ J_1 &= \frac{2}{t}[e^{kL_n-tL_n^2/4}+kJ_0]. \end{aligned} \tag{A.58}$$

Every computed integral is checked positive; cancellation is enclosed by Arb. Every late box has $t_l>0$.

For the reflected term, the combined coefficient of Y is $\Delta-L_n+(1/2+K)\log u$. The checked gates

$$\Delta-L_n+(1/2+K)H<0,\qquad \Delta-(1/2-K)L_n<0 \tag{A.59}$$

justify its height endpoint on the finite and unbounded parts, respectively, even when c_B is negative. Before differentiating the Euler factor, the exact phase multiplier is

$$\frac{\gamma'}{\gamma}-s_A'\log u=\frac{i}{2}\left[\alpha(v)+\alpha(w)-\log u+\frac{t}{2}\{\alpha'(v)(\alpha(v)-\log u)+\alpha'(w)\alpha(w)\}\right].$$

Here the entire source geometry has $|\Re w|,|\Re v|<2$. The sharper elementary bound

$$|\alpha-L/2\mp i\pi/4|\le 5/x+4/x^2\le 5/(x-6) \tag{A.60}$$

follows by adding the logarithm error $|\sigma|/x+\sigma^2/x^2$ and the rational error $3/x$ for $|\sigma|\le 2$. The opposite leading imaginary phases cancel. The phase multiplier is therefore at most $\frac{1}{2}\log(x/(4\pi))-\frac{1}{2}\log u+[5+t_m(\log(x/(4\pi))+1)/2]/(x-6)$. As before, decaying factors are kept with their logarithms in each endpoint reduction.

For $Z_p=e^{t_0\log^2 p/4-\sigma_0\log p}$ set

$$M_E=\prod_p(1+Z_p),\quad L_E=\frac{C_1}{2}\sum_p\frac{\log p\,Z_p}{1-Z_p},\qquad C_1=1+\frac{t_m}{2(x_b-6)}.$$

The program checks $Z_p<1$ before division and

$$L_E+\frac{1}{n}+\frac{5+t_m(\log(x_b/(4\pi))+1)/2}{x_b-6}<\beta_1. \tag{A.61}$$

Thus the full phase weight, including the Euler derivative, is at most $W_N(u)=\log N+\beta_1-\frac{1}{2}\log u$. Its product with the magnitude decreases in $\log N$ when $y_l(L_n+\beta_1-H/2)>1$ on the finite prefix, or $y_l(L_n/2+\beta_1)>1$ on the unbounded part. Both inequalities are checked. They justify the same all-N reduction for the weighted derivative while retaining its negative half-log term.

Explicitly, let $a=c_B+t_0L_n/2$, $g(u)=u^{-a}e^{t_0\log^2 u/4}$, $I_j=I_j(a,t_0;1,M)$, and $W_I=(L_n+\beta_1)I_0-I_1/2>0$. If $a-t_0H/2>0$, set $V_0=1+I_0$ and $V_1=L_n+\beta_1+W_I$. Otherwise use

$$V_0 = 1 + I_0 + g(M), \qquad V_1 = L_n + \beta_1 + W_I + (L_n + \beta_1)\max(1, g(M)). \tag{A.62}$$

To justify these discrete-sum estimates, g has a single valley. The derivative of $(L_n + \beta_1 - \frac{1}{2}\log u)g(u)$ has a concave quadratic sign polynomial in $\log u$. Its positive variation occupies at most one interval, and is bounded by its supremum, at most $(L_n + \beta_1)\max(1, g(M))$. The unit-interval sum-integral identity gives (A.62). In the decreasing case this variation is zero. Maxima are enclosed by $(a + b + |a - b|)/2$, retaining overlapping balls.

Set $J_j = 0$ on a finite band and $J_j = J_j(c_E, t_0; n)$ on the unbounded band. The reflected magnitude and its mollified derivative are bounded by

$$\begin{aligned} A_R &= e^{y(\Delta - L_n)}V_0 + e^{y\Delta}J_0, \\ D_2 &= M_E\{e^{y(\Delta - L_n)}V_1 + e^{y\Delta}(\beta_1 J_0 + J_1/2)\}. \end{aligned} \tag{A.63}$$

The first-polynomial derivative is bounded by $D_1 = C_1(D_B + D_T)/2$. Lemma A.3, multiplied by the modulus bound for all five Euler factors on its complete source disk, gives $D_{\text{rem}} = M_D E_D/(3/20)$. If E_0 denotes the point source error, this proves

$$|\mathcal{F}'| \le D_1 + D_2 + D_{\text{rem}} =: D_{\text{raw}}, \qquad |\mathcal{F} - 1| \le S_B + T_B + M_E(A_R + E_0) =: r_{\text{point}}. \tag{A.64}$$

The first inequality does not require $r_{\text{point}} < 1$. For each time box, take D_{outer} to be the maximum of the outward upper endpoints of all nine wide-rectangle D_{raw} balls. These endpoints are exact dyadic numbers, so their maximum bounds every ball, including overlapping candidates. Cauchy applied to $\mathcal{F}'$ on the physical radius-1/4 disk gives

$$|\mathcal{F}''(z_0)| \le 4D_{\text{outer}}. \tag{A.65}$$

Every center band separately has the checked denominator $m = 1 - r_{\text{point}} > 0$. Put $D_{\text{center}} = D_{\text{raw}}$ there. Let ϵ_1, ϵ_2 be the archimedean first and second jet errors at x_b, using (A.60). With $s_1 = (1 + t_m/[2(x_b - 6)])/2$ and $s_2 = t_m(2/x_b^2 + 24/x_b^3)/8$, direct differentiation gives

$$\left|\left(\frac{\mathcal{E}'}{\mathcal{E}}\right)'\right| \le E_2 = s_2 \sum_p \frac{\log p\, Z_p}{1 - Z_p} + s_1^2 \sum_p \frac{\log^2 p\, Z_p}{(1 - Z_p)^2}.$$

The identities $u = B'/B - \mathcal{E}'/\mathcal{E} + \mathcal{F}'/\mathcal{F}$ and its derivative therefore yield

$$J \le \epsilon_2 + E_2 + \frac{4D_{\text{outer}}}{m} + \left(\frac{D_{\text{center}}}{m}\right)^2 + \left(L_E + \frac{D_{\text{center}}}{m} + \epsilon_1\right)^2. \tag{A.66}$$

Both square terms are required: they come from differentiating $\mathcal{F}'/\mathcal{F}$ and from $|u + \pi/8 + i\Omega|^2$. No wide-circle quotient or center-error/radius substitution occurs.

The program check_late_boundary.py recomputes (A.53)–(A.66) for all 117 boxes and all nine center and wide cutoff bands, importing no stored numerical bounds. It uses the common H7 error $E_{ab} < 3 \cdot 10^{-9}$ and V_7 even on the smaller domain. All 1053 center denominators, all 1053 wide derivative evaluations, and all 1053 final comparisons with (A.49) pass at 512 bits. The smallest final margin is greater than 1.47132047. This proves (A.31) for $9/200 \le t \le T_M$. The early and late intervals share the closed endpoint 9/200. Together they establish (A.31) at every time, discharging the sole remaining boundary premise of Theorem A.6.

A.8. Domains, whole-row use, and reproduction

The dependency order is acyclic. The enlarged approximation supplies the top and infinity estimates and the finite source errors. P2 and Theorem A.5 give the density field and q_M strip simultaneously, using only unsigned force rows. The early and late calculations prove the physical jet boundary directly. Theorem A.6 then propagates it through the band above $b(t)$. Only after these steps are the signed field rows of the reference and final barriers used.

For a row $[t_l, t_r]$ and fixed probe p, the gates $p^2 > q_M(t_l)$, and, where needed, $p \geq 3/5$ and $(p-3/5)^2 \geq q_M(t_l)$ imply $p > h_M(t)$ and $p \geq b(t)$ throughout the closed row. The gates $p \leq 5$ and $t_r \leq T_M$ preserve the stated domains. Equation (A.18) and $\delta < 10^{-5}$ turn $s < f_0(p, t_l) - 10^{-5}$ into the strict field bound $S > s$. Also

$$C(p,t) \leq e^{t_r/100000}[8/25 + A_*/U(p,t_l)] \leq c_{\text{row}}. \tag{A.67}$$

This uses separate monotonic factors and does not assert that C itself decreases in time. These are the whole-row bounds consumed by P8. The uncut density field is used only for $x \geq X$, $h_M(t) < Y \leq 5$, and the uncut jet field only for $x \geq X$, $b(t) \leq Y \leq 5$. No continuation below a pole, to all heights, or left of X is used.

The finite verifications for this appendix are:

Program in support/fields/	Mathematical checks
check_h7.py	Enlarged source, Gaussian/Arias scalars, top and head-center bounds
check_m3a.py	855 initial leaves, 1620 unsigned rows, 1620 nonlocal bottom inequalities
check_fixed_c.py	400 initial cells, supersolution/head constants, nested top disks, 162 boundary rectangles
check_early_boundary.py	45 early boxes, relative Euler jets and complete tails
check_late_boundary.py	117 late boxes, 1053 center and 1053 wide cutoff-band evaluations

Programs write only to an explicitly supplied –output path and reject optimized Python, which would disable their assertions. Late calculations may be divided into adjacent index ranges, but a full verification must cover exactly $45, \ldots, 161$; the early calculation covers exactly $0, \ldots, 44$. The unified command in the main text checks that coverage. The output balls are arithmetic evidence; the reductions in A.1–A.7 give them their continuum and all-cutoff meaning. These are paper proofs with ball-arithmetic verification, and are not asserted to be formalized in Lean.

Appendix B. Source boxes

Each row specifies a closed time-height rectangle on which $V(x, h, t) > L$ for every $x \geq X$. All entries are exact rationals. The printed table is generated from the same catalog consumed by the barrier checker and Lean.

Box	Time interval	Height interval	Floor L
A	[3/50, 601/10000]	[6499/10000, 651/1000]	289/100
01	[601/10000, 303/5000]	[6469/10000, 6501/10000]	721709/250000
02	[303/5000, 611/10000]	[161/250, 809/1250]	2884841/1000000
03	[611/10000, 77/1250]	[641/1000, 1611/2500]	144133/50000
04	[77/1250, 621/10000]	[319/500, 1283/2000]	2880433/1000000
05	[621/10000, 313/5000]	[127/200, 3193/5000]	2878157/1000000
06	[313/5000, 631/10000]	[79/125, 6357/10000]	2875831/1000000
07	[631/10000, 159/2500]	[629/1000, 791/1250]	1436727/500000
08	[159/2500, 641/10000]	[313/500, 3149/5000]	179439/62500
09	[641/10000, 323/5000]	[623/1000, 6269/10000]	2868539/1000000
10	[323/5000, 651/10000]	[12421/20000, 6241/10000]	573541/200000
11	[651/10000, 41/625]	[3091/5000, 497/800]	716347/250000
12	[41/625, 661/10000]	[12307/20000, 773/1250]	1431511/500000
13	[661/10000, 333/5000]	[49/80, 12311/20000]	572121/200000
14	[333/5000, 671/10000]	[12193/20000, 6127/10000]	2858137/1000000
15	[671/10000, 169/2500]	[1517/2500, 12197/20000]	44619/15625
16	[169/2500, 681/10000]	[12079/20000, 607/1000]	35663/12500
17	[681/10000, 343/5000]	[6011/10000, 12083/20000]	2850409/1000000
18	[343/5000, 691/10000]	[2393/4000, 6013/10000]	2847721/1000000
19	[691/10000, 87/1250]	[2977/5000, 11969/20000]	2844973/1000000
20	[87/1250, 701/10000]	[11851/20000, 1489/2500]	568433/200000
21	[701/10000, 353/5000]	[5897/10000, 2371/4000]	567859/200000
22	[353/5000, 711/10000]	[11737/20000, 5899/10000]	70909/25000
23	[711/10000, 179/2500]	[73/125, 11741/20000]	35417/12500
24	[179/2500, 721/10000]	[11623/20000, 2921/5000]	2830293/1000000
25	[721/10000, 363/5000]	[5783/10000, 11627/20000]	565431/200000

The source proof is Proposition C.3. The source evaluator checks these same 26 rectangles; an exact input-identity check connects its box file to this catalog.

Appendix C. Computational estimates

This appendix proves the two numerical inputs called P2 and P7 in the main paper. The accompanying programs recompute their numerical enclosures. They do not infer a theorem from a saved pass flag. The definitions of H_t, S and Ω are those of the main paper. Write

$$X = 5999347341500, \quad X_e = 5900000000000, \quad N(x,t) = \left\lfloor \sqrt{x/(4\pi) + t/16} \right\rfloor.$$

Both calculations use the repaired effective approximation H7, proved with explicit constants in Appendix A. Its derivation includes a uniform vertical Stirling bound and the full Gaussian and Riemann–Siegel remainders. This is the common analytic approximation input for the finite head and the source boxes.

C.1. Normalization and finite sums

For $z = x + iY$, set

$$w = \frac{1 - iz}{2}, \quad v = 1 - w, \qquad \alpha(w) = \frac{1}{2w} + \frac{1}{w-1} + \frac{1}{2}\operatorname{Log}\frac{w}{2\pi},$$
$$M_0(w) = \frac{w(w-1)}{16}\pi^{-w/2}\sqrt{2\pi}\exp\left[\left(\frac{w}{2} - \frac{1}{2}\right)\operatorname{Log}\frac{w}{2} - \frac{w}{2}\right], \quad M_t(w) = M_0(w)e^{t\alpha(w)^2/4}.$$

All logarithms here have their principal branches; their arguments lie far from zero and their cuts on every domain used below. Put

$$\begin{aligned} B_t(z) &= M_t(w), & \gamma &= \frac{M_t(v)}{M_t(w)}, \\ s_B &= w + \frac{t}{2}\alpha(w), & s_A &= v + \frac{t}{2}\alpha(v), \\ P_N(s,t) &= \sum_{n \le N} e^{t\log^2 n/4} n^{-s}. \end{aligned}$$

The source expression is the dimensionless quantity

$$f_t(z) = P_N(s_B, t) + \gamma P_N(s_A, t). \tag{C1}$$

In particular, absolute error allowances for $H_t/B_t - f_t$ must be compared with this expression including its prefactors. B_t is nonzero throughout the source domains.

C.2. A direct certificate for the full vertical boundary

Proposition C.1. For every $0 \le t \le 1/5$ and $0 \le Y \le 1$, $H_t(X + iY) \neq 0$.

C.2.1. A uniform H7 source error

Lemma A.1 gives, for $x \ge X_e$, $0 \le t \le 1/5$ and $0 \le Y \le 7$,

$$|H_t(x+iY)/B_t(x+iY) - f_t(x+iY)| \le E_{ab}(x) + E_c(x,t,Y), \qquad E_{ab}(x) < 3 \cdot 10^{-9}, \tag{C2}$$

where, with $q = x/(4\pi)$ and $L = \log q$,

$$
\begin{aligned}
E_c(x,t,Y) &= q^{-(1+Y)/4}\exp\left[-\frac{tL^2}{16}+\frac{2(3^Y+3^{-Y})}{N(x,t)-1}+10^{-10}+V_7(x)\right],\\
V_7(x) &= \frac{20\sqrt{L^2+\pi^2/4}+200}{x-40}.
\end{aligned}
\tag{C3}
$$

This includes $t=0$: Lemma A.1 proves the closed-time extension by dominated convergence and right-constancy of the actual cutoff. We use the same repaired lemma also on the smaller finite-head height interval $[0,1]$.

For an explicit whole-domain allowance, put $q_e = X_e/(4\pi)$ and $N_e = 685205$. The endpoint check $N_e^2 < q_e$ gives $N(x,t) \geq N_e$. Since $q \geq q_e > 1$, $Y \geq 0$, $t \geq 0$ and V_7 decreases in x, the power is at most $q_e^{-1/4}$ and the heat exponent is nonpositive. The positive function $3^Y + 3^{-Y}$ increases for $Y \geq 0$, so its value at 7 bounds the whole H7 range. Consequently

$$
\begin{aligned}
E_{ab}(x)+E_c(x,t,Y) &< 3\cdot 10^{-9}\\
&\quad + q_e^{-1/4}\exp\left[\frac{2(3^7+3^{-7})}{N_e-1}+10^{-10}+V_7(X_e)\right]\\
&< 0.001215803 < 1/500.
\end{aligned}
\tag{C4}
$$

The head evaluator `bounds.py` recomputes this endpoint expression at 256 bits; `check_h7.py` independently checks the H7 derivation's scalar gates at 512 bits. Taking different extremal heights in the positive inflation and negative power enlarges the error bound, which is valid for all actual points simultaneously. The full vertical-grid acceptance threshold remains $1/500 + 10^{-12}$.

C.2.2. The matrix and its error

At $x = X$ the cutoff is exactly $N = 690950$ throughout $[0, 1/5]$; both strict square-root endpoint inequalities are checked in `bounds.py`. Define

$$
\begin{gathered}
n_0 = 345475, \quad X_m = X + \tfrac{1}{2} = 11998694683001/2, \quad \ell_n = \log(n/n_0), \quad L_0 = \log n_0,\\
M_{e,k} = \sum_{n=1}^{690950} n^{(-1+iX_m)/2}\,\frac{\ell_n^e}{e!}\,\frac{(\ell_n^2/4)^k}{k!}, \qquad 0 \leq e,k < 62.
\end{gathered}
\tag{C5}
$$

`support/head/generate-matrix.c` evaluates precisely these finite sums at 192-bit precision. It forms both factorial recurrences, their outer product, and the complex power for each integer n. Arb retains the errors in the large phase $X_m \log n/2$. No operation-count estimate or discarded-radius argument is used. `regenerate.py` divides $1, \ldots, 690950$ into consecutive disjoint partitions, checks the exact partition endpoints, adds their balls at 256 bits, and verifies enclosure under the final serialization. The distributed `data/matrix.txt` is the result of this complete regeneration.

For $s = (1 - Y + iX)/2$, put $w = 1 - s$, $s_c = \bar{s}$ and

$$
b_B = -Y/2 - i/4, \quad b_A = Y/2 - i/4, \qquad v_B = b_B - \frac{t}{2}(\alpha(w) - L_0), \quad v_A = b_A - \frac{t}{2}(\alpha(s_c) - L_0),
$$

$$
\begin{aligned}
p_B &= \exp\left[L_0\left(b_B - \frac{t}{4}(2\alpha(w) - L_0)\right)\right],\\
p_A &= \exp\left[L_0\left(b_A - \frac{t}{4}(2\alpha(s_c) - L_0)\right)\right],\\
F_e(t) &= \sum_{k<62} M_{e,k} t^k.
\end{aligned}
$$

The evaluated polynomial is

$$
F_M(t,Y) = p_B \sum_{e<62} F_e(t) v_B^e + \gamma \overline{p_A \sum_{e<62} F_e(t) v_A^e}. \tag{C6}
$$

Writing $\log n = L_0 + \ell_n$ shows term by term that replacing both Taylor polynomials in (C6) by their exponential series gives exactly (C1). The imaginary constant $-1/4$ is $(X - X_m)/2$. The factorials occur in (C5) once, and are not inserted again during Horner evaluation. `matrix.py` evaluates $\log M_0$ directly, so the gamma ratio has the same normalization as (C1).

The whole box $0 \le t \le 1/5$, $0 \le Y \le 1$ satisfies

$$
|v_A|, |v_B| < 33/50, \qquad \Re\alpha(w), \Re\alpha(s_c) > L_0.
$$

These inequalities are checked on one enclosing complex ball box. At actual points, the last inequalities imply $|p_B| \le 1$ and $|p_A| \le n_0^{Y/2} \le \sqrt{n_0} < 600$. The H7 gamma bound (A.3) gives $|\gamma| \le \exp[Y(10^{-11} - \frac{1}{2}\log(X/(4\pi)))] \le 1$, including equality when $Y = 0$.

Set $U = L_0^2/20$ and $W = (33/50)L_0$. Since $U, W < 63$, the elementary geometric factorial-tail bound is

$$
R(a) = \frac{a^{62}}{62!\,(1 - a/63)} \quad (0 \le a < 63).
$$

The error in a product of the two truncated exponentials is at most $R(U)e^W + e^U R(W)$. Since $|\ell_n| \le L_0$ and $\sum_{n \le N} n^{-1/2} \le 2\sqrt{N} - 1$, we obtain

$$
|f_t(X + iY) - F_M(t,Y)| \le 1200(2\sqrt{N} - 1)\{R(U)e^W + e^U R(W)\} < 6.57 \cdot 10^{-19} < 10^{-12}. \tag{C7}
$$

The coefficient balls already contain their generation and serialization errors. Equation (C7) accounts only for the omitted analytic series.

C.2.3. Coverage and nonvanishing

The exact nominal cells are

$$
I_i = [i/2000, (i+1)/2000], \quad 0 \le i < 400, \qquad J_j = [j/2000, (j+1)/2000], \quad 0 \le j < 2000.
$$

Their Cartesian products cover the entire closed parameter rectangle. `support/head/verify.py` evaluates every pair (i, j) at 192 bits and checks that the evaluation ball for each parameter contains both nominal rational endpoints. Enclosing parameter intervals can overlap or extend slightly beyond the nominal domain; the error estimates are applied only to actual points of the target rectangle. For a fixed time row the same interval polynomials $F_e(I_i)$ are reused, which is an exact algebraic regrouping. All 800000 cells satisfy the certain inequality

$$|F_M(I_i, J_j)| > 1/500 + 10^{-12}. \tag{C8}$$

The smallest certified lower endpoint in the recorded replay is greater than 1.2765849605; this diagnostic minimum is not substituted for the individual gates. Equations (C4), (C7) and (C8) show $H_t(X+iY)/B_t(X+iY) \neq 0$. This proves Proposition C.1.

C.2.4. The finite head remains real

Proposition C.2 (P2). Every zero ρ of H_t with $|\Re\rho| \leq X$ is real, for every $0 \leq t \leq 1/5$.

At $t = 0$, Platt–Trudgian, Theorem 1 covers $0 < x \leq X$, since $X/2 = 2999673670750 < 3 \cdot 10^{12}$; evenness covers $x < 0$. At $x = 0$, positivity of $\Phi(u)$ and $\cosh(Yu)$ gives $H_0(iY) > 0$. Thus the zero-height endpoint is covered too. The finite-RH theorem is cited, not recomputed.

Proposition C.1, conjugation and evenness make both sides $\Re z = \pm X$ zero-free for $|\Im z| \leq 1$ throughout the time interval. At time zero the horizontal sides of the rectangle $K = \{|\Re z| \leq X,\ |\Im z| \leq 1\}$ are zero-free by the finite-RH theorem. For $t > 0$, de Bruijn's strip contraction places all zeros strictly inside $|\Im z| < 1$. Thus the entire boundary of K is zero-free for all these times. By continuity and the argument principle, the number of enclosed zeros with multiplicity is constant.

The times at which all enclosed zeros are real form a closed set: a nonreal zero at a limiting time has a small disk away from the real axis with a zero-free boundary, and Rouché's theorem would force a nonreal zero at nearby times. The set is also open to the right. A simple real zero stays real by conjugation and local uniqueness. At an m-fold real zero, the forward Hermite splitting in Polymath, Proposition 3.1(ii) gives m zeros in disks of radius $O(\epsilon)$ around distinct real centres separated by order $\sqrt{\epsilon}$, at time $t+\epsilon$. For sufficiently small $\epsilon > 0$ these disks are disjoint and invariant under conjugation; each contains exactly one zero counting multiplicity, hence that zero is real. There are finitely many clusters inside K, and no boundary entry is possible, so one right-neighbourhood works for all of them. Starting at zero, closedness and this extension property prove the assertion through 1/5. This conclusion concerns the finite strip $|\Re\rho| \leq X$.

C.3. A uniform source recipe independent of the jet field

For each box $[t_-, t_+] \times [h_-, h_+]$ in Appendix B, define $Y_j = (j+2)h$, $j = 1, 2, 3$, and

$$V = \frac{15}{14} S(x, 3h, t) - \frac{16}{21} S(x, 4h, t) + \frac{1}{6} S(x, 5h, t).$$

Proposition C.3 (P7). At every point of each of these 26 closed boxes, for every $x \geq X$, all three logarithmic derivatives are defined and V is strictly greater than the corresponding stated rational floor.

The proof uses only the H7 source and elementary zeta estimates below. In particular it has no input from the old barrier, the jet field, or a selected contour's actual zero height. Write

$$N_* = 690950, \quad L_* = \log N_*, \quad r = 1/20,$$
$$T_m = 1/5, \quad \beta = 1/2, \quad r_1 = 500001/1000000, \quad \Omega_L = 6722911/10^6.$$

The 384-bit program `support/source/verify.py` evaluates the formulas below for every box and rejects unless all the indicated inequalities hold.

C.3.1. Uniform domains and source constants

Every radius-r disk about a source centre has real part at least $X - r > X_e$ and height strictly between 0 and 7. The program verifies $X - r > 4\pi N_*^2$. Thus every actual cutoff on every disk is at least N_*. The square root defining $N(x,t)$ has derivative at most $1/(8\pi N_*)$; its variation on a disk is at most $r/(8\pi N_*) < 1$. The disk and centre cutoffs therefore differ by at most one.

At each centre, freeze its actual cutoff only for the local holomorphic germ

$$E_{N_c}(z) = H_t(z)/B_t(z) - P_{N_c}(s_B, t) - \gamma P_{N_c}(s_A, t).$$

The centre cutoffs at different x, t can differ. No floor function is differentiated. H7 and the two changed-cutoff terms will bound E_{N_c} on the whole disk. All Cauchy estimates below apply to this analytic remainder, not to a quotient by H_t.

Set

$$\begin{aligned} \ell_{\text{cut}} &= \log\left(1 - \frac{T_m}{16N_*^2}\right), & c_{\text{cap}} &= -\frac{T_m \ell_{\text{cut}}}{4}, \\ c_{\text{corr}} &= \frac{T_m}{X_e^2}, & d_\gamma &= 10^{-11} - \ell_{\text{cut}}/2, \\ k &= \frac{T_m}{2(4\pi N_*^2 - \pi T_m/4 - 6)}. \end{aligned}$$

Combining the H7 real-part, gamma and kappa bounds with the actual cutoff relation gives, for $L = \log N$,

$$\begin{aligned} c(Y) &= (1+Y)/2 - c_{\text{cap}} - c_{\text{corr}}, \\ \Re s_B &\geq c(Y) + tL/2, \\ \Re s_A &\geq c(Y) - Y - kY + tL/2, \\ |\gamma| &\leq e^{d_\gamma Y} N^{-Y}. \end{aligned} \tag{C9}$$

The cutoff relation is used before replacing N by N_*; in particular $\log(x/(4\pi)) \geq 2\log N + \log(1 - T_m/(16N^2))$. For $y_j^- = (j+2)h_-$, $y_j^+ = (j+2)h_+$, put

$$c_j = c(y_j^-), \qquad a_j = c_j + (t_-/2)L_*. \quad \text{Then} \quad a_j = a_1 + (j-1)h_-/2. \tag{C10}$$

Every actual $s_B(x + iY_j, t)$ and every ideal shifted argument $s_B(x + 3ih, t) + (j-1)h/2$ has real part at least a_j. The program checks

$$c_j > 1, \quad c_j > 5/\log 1024, \quad a_j - r_1 r > 1, \quad a_j > 3/L_*. \tag{C11}$$

Let $L_x = \log(x/(4\pi))$ and define

$$\begin{aligned} d_a(x) = \frac{8}{x-20}, \quad A_0(x) = \tfrac{1}{2}\sqrt{L_x^2 + \pi^2/4} + d_a(x), \quad A_1(x) = 1/x + 6/x^2, \\ b_B = d_a(X_e)/2 + T_m A_0(X_e) A_1(X_e)/4, \qquad \eta = t_+/[4(X_e - 6)]. \end{aligned}$$

The H7 archimedean estimates imply $|B_t'/B_t+\pi/8+i\Omega(x)| \le b_B$ and $|(s_B)_z+i/2|, |(s_A)_z-i/2| \le \eta$. The products A_0A_1 and $A_1 \log x$ decrease for $x \ge X_e$: $A_0' \le 1/(2x)$, $A_0 > 1/2$, and differentiation of the inverse powers gives the signs. The code checks $b_B < 10^{-10}$, $1/2+\eta < r_1$, and the reflected phase allowance

$$\frac{1}{N_*} + d_a(X_e) + \frac{t_+}{2} A_0(X_e)A_1(X_e) + \frac{t_+}{4} A_1(X_e)\log X_e < \beta. \tag{C12}$$

Together with (C11), the spatial rate bound places the whole zeta reference disk in $\Re s > 1$.

C.3.2. The finite heat remainder and its infinite majorant

The identity at the actual cutoff is

$$P_N(s_B,t) = \zeta(s_B) + \frac{t}{4}\zeta''(s_B) + R_{\text{heat}} - T_N,$$
$$R_{\text{heat}} = \sum_{n\le N} (e^{t\log^2 n/4} - 1 - t\log^2 n/4)n^{-s_B}, \quad T_N = \sum_{n>N} (1+t\log^2 n/4)n^{-s_B}. \tag{C13}$$

Only the finite sum is heated exponentially. No infinite positive-time heated Dirichlet series is introduced. For $v = \log n$ and $H(u) = 1-(1+u)e^{-u}$, the included summands in R_{heat} are bounded by

$$w_j(n) = n^{-c_j} \exp\{t_-[v^2/4 - v\max(L_*,v)/2]\}\, H(t_+v^2/4). \tag{C14}$$

Indeed $L \ge \max(L_*,v)$ for included n, the combined heat/source exponent is nonpositive and uses t_-, while H increases and uses t_+. After this combination the positive majorant can be extended to all $n \ge 2$. Write $H_{j,k} = \sum_{n\ge 2} (\log n)^k w_j(n)$ for $k = 0,1$.

Since $H(u) = \int_0^u ve^{-v}dv \ge u^2e^{-u}/2$, $H'/H \le 2/u$ for $u > 0$. The logarithmic derivatives of the summands for $n \ge 1024$ are at most $-c_j + (k+4)/\log n < 0$ by (C11). The head $2 \le n \le 1024$ is summed directly and the remainder is bounded by the integral from 1024 to infinity. In logarithmic coordinates its decaying factor is

$$D_j(v) = \exp\{-(c_j-1)v + t_-[v^2/4 - v\max(L_*,v)/2]\}.$$

For each of the 1024 consecutive cells of width 1/16 beginning at $\log 1024$, evaluate D_j at the left endpoint and the increasing factor $v^kH(t_+v^2/4)$ at the right. Above $V_* = \log 1024 + 64 > L_*$, put $q_j = c_j - 1 + t_-V_*/2 > 0$ and $P_j = e^{-(c_j-1)V_* - t_-V_*^2/4}$. A tangent exponential bound gives the remaining tails

$$P_j/q_j, \qquad P_j(V_*/q_j + 1/q_j^2). \tag{C15}$$

For the ordinary tail define the exact integral

$$T_k(a) = N_*^{1-a} \sum_{\ell=0}^{k} \frac{k!}{(k-\ell)!} \frac{L_*^{k-\ell}}{(a-1)^{\ell+1}} = \int_{N_*}^{\infty} (\log u)^k u^{-a}\, du.$$

The decreasing-summand gates in (C11) show that the value and first logarithmic-moment tails of T_N are bounded by

$$T_0(a_j) + (t_+/4)T_2(a_j), \qquad T_1(a_j) + (t_+/4)T_3(a_j). \tag{C16}$$

The physical derivative of these terms and the heat remainder costs an additional factor at most r_1 on their first logarithmic moments.

C.3.3. Reflected terms for every actual cutoff

For an actual included term, (C9) gives the reflected amplitude majorant

$$e^{d_\gamma Y - YL} n^{-c(Y)+Y+kY} \exp\{t(v^2/4 - Lv/2)\}. \tag{C17}$$

Its logarithmic derivative in Y is bounded above by $d_\gamma - (1/2-k)L_* < 0$, an explicit checked gate. This gate also implies $k < 1/2$. The heat exponent is nonpositive. Thus the full expression is bounded by its corner $Y = y_j^-$, $t = t\_$. Writing $y = y_j^-$ and $b = c_j - y - ky$, the resulting envelope is $e^{d_\gamma y - yL} n^{-b} e^{t_-(v^2/4 - Lv/2)}$.

For the derivative, H7 gives the raw phase identity, with $g = \log\gamma$,

$$|g_z - (s_A)_z \log n - i(L - \tfrac{1}{2}\log n)| \le 1/N + d_a(x) + (t/2)A_0(x)A_1(x) + (t/4)A_1(x)\log x.$$

By (C12), the derivative is bounded by $(L+\beta)$ times the amplitude envelope. This retains the cancellation between the gamma and index phases.

Here is the all-N reduction. The summand as a function of real $u \in [1, N]$ has at most one minimum, since its logarithmic derivative in $\log u$ increases. Its sum is consequently bounded by both endpoint terms plus the integral. Set

$$I(L) = \int_0^L e^{(1-b-t\_L/2)v + t\_v^2/4}\,dv, \quad F(L) = e^{d_\gamma y - yL}[1 + I(L) + e^{-bL - t\_L^2/4}].$$

Let $a_0 = 1 - b > 0$, $e = b + y$, and $K_0 = (1 - e^{-a_0 L_*})/a_0$. The program checks

$$(y - 1/(L_* + \beta))K_0 > 1, \qquad e(L_* + \beta) > 1. \tag{C18}$$

For $A(L) = e^{-yL}I(L)$ and $E(L) = e^{(1-b-y)L - t\_L^2/4}$, differentiation gives $A' \le E - yA$. Substitution $v = L - w$ gives $A/E = \int_0^L e^{-a_0 w + t\_w^2/4}dw \ge K_0$ for $L \ge L_*$. Therefore $[(L+\beta)A]'$ is negative by (C18). Both normalized endpoint terms, with the same factor $L+\beta$, also decrease: the $n = N$ term uses the second gate of (C18), and the $n = 1$ term uses the first. Hence $(L+\beta)F(L)$ and $F(L)$ decrease for every real $L \ge L_*$.

At L_* the additional gate $1 - b - t\_L_*/2 > 0$ makes the integrand increasing. The program bounds $I(L_*)$ by 1024 right-endpoint rectangles and obtains upper bounds $\mathrm{Ref}_{j,0} = F^+(L_*)$ and $\mathrm{Ref}_{j,1} = (L_* + \beta)F^+(L_*)$. The two endpoint terms are included in both. This proves the cutoff reduction on the entire unbounded range; it is not a sample at N_*.

C.3.4. Fixed-cutoff source disks and both cutoff changes

Let $y_d = y_j^- - r$, $y_u = y_j^+ + r$, $L_e = \log(X_e/(4\pi))$ and

$$I_+ = \frac{2(3^{y_u} + 3^{-y_u})}{685205 - 1} + 10^{-10} + \frac{20\sqrt{L_e^2 + \pi^2/4} + 200}{X_e - 40}.$$

The H7 point-error contribution other than its $3 \cdot 10^{-9}$ allowance is bounded at a centre and on its disk, respectively, by

$$E_c = e^{-(1+y_j^-)L_e/4 - t\_L_e^2/16 + I_+}, \qquad E_d = e^{-(1+y_d)L_e/4 - t\_L_e^2/16 + I_+}. \tag{C19}$$

All negative factors use lower endpoints; the positive height inflation uses y_u. The unbounded-x reduction uses the decreasing functions in H7.

At a disk point with actual cutoff $m \geq N_*$, a changed term has index $n = m$ or $m + 1$. Put $L = \log m$ and $\delta = \log(n/m) \in [0, \log(1 + 1/m)]$. Its primary exponent is at most

$$-a_d(L + \delta) - tL^2/4 + t\delta^2/4, \qquad a_d = (1 + y_d)/2 - c_{\text{cap}} - c_{\text{corr}} > 0.$$

Dropping $-a_d\delta$, bounding the positive correction by $T_m \log(m + 1)/(2m)$ and using decrease in m gives

$$\text{Cut}_1 = \exp\left[-a_d L_* - t\_L_*^2/4 + \frac{T_m \log(N_* + 1)}{2N_*}\right]. \tag{C20}$$

The reflected changed term is at most $\text{Cut}_1 G_{\text{cut}}$, where

$$G_{\text{cut}} = \exp\{d_\gamma y_u + y_u \log(1 + 1/N_*) + k y_u \log(N_* + 1)\}. \tag{C21}$$

Here the gamma/index ratio is combined before taking endpoints. In particular the positive product $\log(m + 1)T_m/[2(4\pi m^2 - \pi T_m/4 - 6)]$ decreases for $m \geq N_*$, as direct differentiation shows. An isolated unbounded $\log m$ is not replaced by $\log N_*$.

The source error has no cutoff discrepancy at the centre. On the disk it includes both changed terms. Cauchy's estimate on the fixed-N_c germ gives

$$\text{Src}_{j,0} = 3 \cdot 10^{-9} + E_c, \qquad \text{Src}_{j,1} = \{3 \cdot 10^{-9} + E_d + \text{Cut}_1(1 + G_{\text{cut}})\}/r. \tag{C22}$$

Only the first derivative is needed. The source circle can cross $x = X$; its validity comes from H7 on $x \geq X_e$, without a field assumption.

C.3.5. Nonvanishing and a single logarithmic-remainder bound

Let $\text{Tail}_{j,k}$ denote (C16). At each actual point, (C13) and (C1) yield

$$\begin{gathered} G := H_t/B_t = \zeta(s_B)(1 + w + e), \quad w = (t/4)Z(s_B), \quad Z = \zeta''/\zeta, \quad e = R/\zeta(s_B), \\ |R| \leq R_0 = H_{j,0} + \text{Tail}_{j,0} + \text{Ref}_{j,0} + \text{Src}_{j,0}, \\ |R_z| \leq R_1 = r_1(H_{j,1} + \text{Tail}_{j,1}) + \text{Ref}_{j,1} + \text{Src}_{j,1}. \end{gathered} \tag{C23}$$

For real $a > 1$, define

$$W_k(a) = \sum_{n \geq 2} \Lambda(n)(\log n)^k n^{-a}, \qquad m(a) = \zeta(2a)/\zeta(a) > 0,$$

where $\Lambda(n)$ is the von Mangoldt function. The Euler product implies $|\zeta(s)| \geq m(a)$ for $\Re s \geq a$. The real zeta Taylor coefficients at a_j give certified enclosures of $W_0, \dots, W_3$. Put

$$Z_0 = W_1 + W_0^2, \quad Z_1 = W_2 + 2W_0W_1, \quad Z_2 = W_3 + 2W_1^2 + 2W_0W_2,$$
$$V_0^+(a) = W_0 + (t_+/4)Z_1, \quad V_1^+(a) = W_1 + (t_+/4)Z_2,$$
$$e_0 = R_0/m(a_j), \quad e_1 = (R_1 + r_1W_0R_0)/m(a_j),$$
$$w_0 = (t_+/4)Z_0, \quad w_1 = (t_+/4)r_1Z_1, \quad \rho_0 = w_0 + e_0.$$

Every one of the 78 bands passes the strict gate $\rho_0 < 1$. Consequently $|1 + w + e| \geq 1 - \rho_0 > 0$ at every centre, proving nonvanishing of G and hence of H_t there. This argument requires no prior zero-free region for H_t on the source disks.

For $\ell = \log\zeta$, the exact logarithmic-derivative remainder is

$$\mathcal{E} = \frac{G_z}{G} - (s_B)_z[\ell'(s_B) + (t/4)Z'(s_B)] = \frac{e_z - w_z(w + e)}{1 + w + e}.$$

Thus throughout the box

$$|\mathcal{E}| \leq E_j := \frac{e_1 + w_1\rho_0}{1 - \rho_0}. \tag{C24}$$

This is the sole normalization route used by the publication checker.

C.3.6. The common-phase combination and the negative middle coefficient

At the actual time, the function

$$\Phi_t(s) = -\ell'(s) - (t/4)Z'(s) = \sum_{n\geq 2} A_t(n)n^{-s}$$

has nonnegative coefficients

$$A_t(n) = \Lambda(n) + (t/4)[\Lambda(n)\log^2 n + \log n \sum_{ab=n} \Lambda(a)\Lambda(b)].$$

This ordinary Dirichlet series converges absolutely for $\Re s > 1$. Write $A = 15/14$, $B = 16/21$, $D = 1/6$ and $\mu = A - B + D = 10/21$. For a shared argument s_1 and $v = n^{-h/2}$, the combined coefficient is $P(v) = A - Bv + Dv^2 > 0$ on $[0, 1]$. Moreover

$$\frac{3A - 4Bv + 5Dv^2}{2} = \frac{135 - 128v + 35v^2}{84} \geq \frac{1}{2}.$$

Hence $n^{-3h/2}P(n^{-h/2})$ decreases in h. First increase the positive coefficients from t to t_+; then use the lower real shift from t_-; finally combine the entire height factor before taking $h = h_-$. This proves the coherent upper bound

$$\Re\sum_{j=1}^{3}(A, -B, D)_j\Phi_t(s_1 + (j-1)h/2) \leq AV_0^+(a_1) - BV_0^+(a_2) + DV_0^+(a_3). \tag{C25}$$

The three real zeta quantities on the right are values of one exact expression. The program evaluates it by signed interval arithmetic, so the subtracted term uses the correct enclosure direction. It does not subtract three unrelated upper bounds. The factored common base exponent need not itself exceed one; no zeta series at that incomplete exponent is used.

The actual arguments differ from the ideal ones by at most

$$\epsilon_j = t_+(j-1)h_+/[4(X_e-6)].$$

Both endpoints, and therefore their joining segment, have real part at least a_j. The associated derivative allowance is $V_1^+(a_j)$. The density identity from the definition of Φ_t is $S_j = \Omega - \frac{1}{2}\Re\Phi_t(s_j) + \text{error}$, with error modulus at most $b_B + \eta V_0^+(a_j) + E_j$ by (C24).

Finally $\Omega(x) > \Omega_L$ for $x \geq X$; the program checks it at the smaller $X - 10^6$. Since $\mu > 0$, only this lower bound on the unbounded archimedean term is needed. We conclude

$$\begin{aligned} V \geq L_{\text{box}} := \mu\Omega_L &- \tfrac{1}{2}[AV_0^+(a_1) - BV_0^+(a_2) + DV_0^+(a_3)] \\ &- \tfrac{1}{2}[B\epsilon_2 V_1^+(a_2) + D\epsilon_3 V_1^+(a_3)] - 2b_B \\ &- \eta[AV_0^+(a_1) + BV_0^+(a_2) + DV_0^+(a_3)] - [AE_1 + BE_2 + DE_3]. \end{aligned} \tag{C26}$$

For each of the 26 boxes the computed ball for this exact lower expression lies strictly above its published rational floor. For example, the A-box lower expression exceeds $2.8911909440 > 289/100$, and the box-25 expression exceeds $2.8271559938 > 565431/200000$. Every box, including all its edges and corners, is checked separately. This proves Proposition C.3. The original larger screening target on some boxes is irrelevant to this strict floor comparison, and its failure was not interpreted as an upper bound for V.

C.4. Reproduction and trust boundary

From the publication root, using the pinned Python environment:

```
python support/head/verify.py
python support/head/spotcheck.py
python support/source/verify.py
```

The first run consumes the stored coefficient balls but computes every boundary cell anew; the second computes every source enclosure from the exact box parameters and the formulas above. Both refuse failed or ambiguous inequalities. Optional `--output NEW.json` preserves a fresh replay record; the default writes no files.

The additional `spotcheck.py` command independently evaluates both original finite sums at $(t, Y) = (0, 0)$ and $(1/5, 1)$, at 256 bits. Its direct result agrees with (C6) within $2.81 \cdot 10^{-33}$ at both points. This checks the factorization and normalization through a different evaluation path. Its scope is two points, and the full-domain proof remains (C7)–(C8).

To remove dependence on the stored matrix:

```
python support/head/regenerate.py --jobs 8 --output /tmp/dbn-matrix-new.txt
python support/head/verify.py --matrix /tmp/dbn-matrix-new.txt
```

The regeneration has no input numerical table. Exact matrix-centre and partition metadata are checked. Runtime measurements are included in the replay records and depend on the machine and execution profile. The programs, rational input boxes, full matrix balls, component records and environment details are distributed under `support/head/` and `support/source/`.

The analytic reductions in this appendix, H7, the published finite-RH theorem, and the mathematical behavior of Arb/FLINT are the trust boundary. The C regeneration also trusts its compiler and linked libraries. These are computer-assisted mathematical proofs conditional on the stated literature and software foundations; the numerical checkers do not constitute

Lean kernel verification of those foundations. The quoted intervals and error bounds apply to their exact stated domains and cutoffs.

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[Lean] The Lean community, Lean 4 and mathlib, versions 4.33.1. Exact revisions are recorded in the companion project's Lake manifest.

[LC] leancert, revision 4ea18bb66dfcd7f18f260bc25302400f5a3cafd7. The companion imports its heat-flow analysis and uses its kernel-mode interval evaluator. Imported definitions are connected to the paper's heat integral by proved equalities.